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Detour Domination Cover Edge Pebbling Number in Graphs and its Algorithm

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Abstract

Given a connected graph G, an edge pebbling move involves the removal of two pebbles from one edge and the addition of one pebble to an adjacent edge. In this paper, we introduce the concept of detour domination cover edge pebbling number and find out the detour domination cover edge pebbling number for some standard graphs. We introduce a set called extended edge-to-edge detour set. We carry out the edge pebbling move in the concepts of extended edge-to-edge detour set and cover edge pebbling number to arrive a new graph invariant called the detour domination cover edge pebbling number. The detour domination cover edge pebbling number, $\psi_e^*(G)$, is the minimum number of pebbles required to cover all the edges in the extended edge-to-edge detour set with a pebble, and that set mostly form an edge dominating set on its own whereas the minimum number of edges are occasionally added to the set to achieve edge domination, for the initial configuration of pebbles can be altered by a sequence of pebbling moves. Also, we give an example and an algorithm that offer grounds for the results.

Key words: Edge pebbling move, Edge dominating set, Edge detour path, Extended edge-to-edge detour set, Detour domination cover edge pebbling number.

2020 Mathematics Subject Classification: 05C12, 05C38, 05C57, 05C70, 05C85.

1 Introduction

Graph pebbling is an emerging concept in Graph theory and in which underlying fundamentals have been studied extensively. Introduced by F.R.K. Chung [4] in 1989, the concept has been explored in-depth by various authors, including G. Hurlbert [8] whose

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groundbreaking work has significantly advanced the understanding of Graph Pebbling. A pebbling move involves removal of two pebbles from a vertex and placement of one pebble on an adjacent vertex, while the other pebble will be discarded from the play. The pebbling number, $f(G)$, [4] represents the minimum number of pebbles required to reach any target vertex with one pebble, regardless of the initial configuration and the pebbling problem analogous to transmitting data from a source to a destination.

Several types of pebbling numbers have been investigated, as referenced in [2, 7, 9, 18]. In 2020, P. Paul expanded the concept by introducing edge pebbling numbers and cover edge pebbling numbers for various graph types. An edge pebbling number, $P_E(G)$, [14] is defined to be the minimum number of pebbles needed to move one pebble to any specified edge. The cover edge pebbling, $CP_E(G)$, [14] aims to cover all edges with at least one pebble, regardless of their initial placement. P. Paul has also brought forth the idea of cover edge pebbling number for complete graph, n-star graph and friendship graph and she has also made significant introductions to the field of covering cover edge pebbling number in graphs. Those seeking a deeper understanding of these concepts are encouraged to explore [15, 16, 17].

Research on distances in graphs is extensive, with numerous applications in real-life problems. In particular, Euclidean distance [5] is used in Machine Learning Algorithms. This paper focuses on detour distance in graphs. Chartrand et. al. [3] examined the basic concepts of detour distance in graphs.

Lourdusamy et al. [11] have precisely established the detour pebbling number, $f^*(G)$, for various standard graphs, see [10, 12]. The concept of domination cover pebbling, $\psi(G)$, was pioneered by J. Gardner et al. [6] and they have also presented the domination cover pebbling number for complete graphs, wheel graphs, and for some standard graphs. Additional information about domination cover pebbling can be found in [22]. Vincylin et al. [19, 20, 21] have introduced the detour edge pebbling number, $f_e^*(G)$, detour edge cover pebbling number, $f_{ec}^*(G)$, and domination cover edge pebbling number, $\psi_e(G)$, for various graphs.

In this paper, we introduce a set called extended edge-to-edge detour set and we define the detour domination cover edge pebbling number and compute the detour domination cover edge pebbling number for the wheel graph, path graph, comb graph and cycle graph.

An edge set $S \subseteq E(G)$ is called an extended edge-to-edge detour set of a graph G if it contains two edges whose edge detour distance is maximum in G, and every other edge in the set S does not lie on the edge detour path joining these two edges.

The detour domination cover edge pebbling number, $\psi_e^*(G)$, is defined as the minimum number of pebbles required so that, from any initial pebble configuration, it is possible to place at least one pebble on every edge of an extended edge-to-edge detour set through a

sequence of edge pebbling moves. This set typically forms an edge-dominating set however in certain cases, additional edges may be included to ensure edge domination.
We give an example in the next section that paves the way for finding out the detour domination cover pebbling number for any graph and we give an algorithm in the end.

2 Preliminaries

Definition 2.1. [1] A set $S \subseteq E(G)$ is called an edge-to-edge detour set for a graph G if every edge of G is an element of S or lies on a detour joining a pair of edges of S.
Definition 2.2. [13] A set D of edges is called an edge dominating set for a graph G if every edge not in D has at least one neighbor in D.
Definition 2.3. [19] An edge detour path between e_x and e_y is a sequence of edges P_e^* $= \{e_x,\ e_1,\ e_2,\ ...,\ e_k,\ e_y\}$ such that $e_i \cap e_{i+1} \neq \emptyset$, no edge appears more than once in the sequence and the path allows repeated vertices. Therefore, the edge detour distance between the edges e_x and e_y, $d^*(e_x,\ e_y) = |P_e^*|$-1 where $|P_e^*|$ is the total number of edges in the edge detour path.
Theorem 2.4. [11] For a path P_n on n vertices, the detour pebbling number $f^*(P_n) = 2^{n-1}$
Theorem 2.5. [11] For a cycle C_n on n vertices, the detour pebbling number $f^*(C_n) = 2^{n-1}$
Theorem 2.6. [11] For a wheel graph W_n on n+1 vertices, the detour pebbling number $f^*(W_n) = 2^n$
Theorem 2.7. [14] For a path P_n, the cover edge pebbling number $CP_E(P_n) = 2^{n-1}$-1
Theorem 2.8. [14] For a cycle C_n, the cover edge pebbling number $CP_E(C_n) =$

$$\begin{cases} 3(2^{\frac{n}{2}}\text{-}1), \text{ if } n \text{ is even} \\ 2^{\frac{n+3}{2}}\text{-}3, \text{ if } n \text{ is odd.} \end{cases}$$

Theorem 2.9. [6] For a path P_n, the domination cover pebbling number $\psi(P_n) = \lceil \frac{2^{n+1}-2}{7} \rceil$
Theorem 2.10. [6] For a cycle C_n, the domination cover pebbling number $\psi(C_n) =$

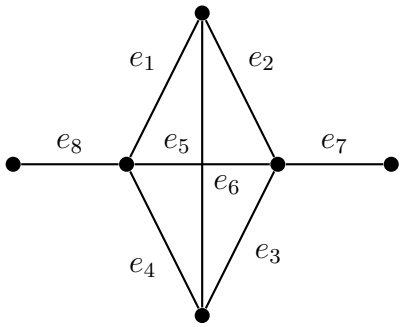
$$\begin{cases} \lfloor \frac{2^{k+1}-1}{7} \rfloor, \text{ if } n = 2k\text{+}1,\ k \geq 1 \\ \lfloor \frac{3(2^{k+1})-3}{7} \rfloor, \text{ if } n = 2k,\ k \geq 2. \end{cases}$$

Theorem 2.11. [6] For a wheel W_n, the domination cover pebbling number $\psi(W_n) = n$-2
Notation 2.12. Throughout this paper, we denote:

- EEED as the extended edge-to-edge detour set.

- DDCEPN as the detour domination cover edge pebbling number.
- $P(e_1) = 1$ as the pebble on the edge e_1 is 1.
- $\psi_e^*(G)$ as the detour domination cover edge pebbling number of a graph G.
- $E(G)$ as the edge set of a graph G.
- $S(W_n)$ as the spokes of wheel graph W_n.
- $C(W_n)$ as the cycle edges of wheel graph W_n.

Example 2.13.



A simple connected graph G.

Consider the above graph G. Here, the maximum edge detour distance occurs between the edges e_7 and e_8 and the edge detour path is e_7, e_2, e_1, e_4, e_3, e_5, e_8. So, the possible extended edge-to-edge detour set is S_1: e_7, e_8, e_6 and these edges also form an edge dominating set.

Now, we need to check on the number of pebbles required to cover all the edges in the set S_1.

If $P(e_5) = 8$ and consider the distribution of 1 pebble move to the edges e_7, e_8 and e_6, we can cover all the edges of S_1. So, 8 pebbles are needed in this distribution. But, if $P(e_8) = 8$, any one of the edges of S_1 will be left uncovered. So, 8 is not the required number of pebbles.

If $P(e_8) = 9$, all the edges of S_1 can be covered but if $P(e_6) = 8$ and $P(e_5) = 1$, we cannot cover all the edges of S_1. So, 9 is not the required number of pebbles.

If $P(e_1) = 8$, $P(e_3) = 1$ and $P(e_4) = 1$, we can cover all the edges of S_1 with 10 pebbles but if $P(e_7) = 8$, $P(e_1) = 1$ and $P(e_4) = 1$ and consider the distribution of 1 pebble move to the edges e_6 and e_8, the edge e_7 will be left uncovered. So, 10 is not the required number of pebbles.

If $P(e_7) = 9$, $P(e_1) = 1$ and $P(e_4) = 1$, we can cover all the edges of S_1 with 11 pebbles.

So, 11 pebbles are needed in this distribution.
Similarly, the pebbles may be placed on any edge of G. However, we stop the process here since placing 9 pebbles on the edge e_7 is the maximum number of pebbles required to cover all the edges in the set S_1 when all the pebbles are initially placed on a single edge of G. Furthermore, if we modify the pebble configuration, then 11 pebbles are sufficient to cover all the edges in the set S_1.
Thus, the detour domination cover edge pebbling number for the graph G is $\psi_e^*(G) = 11$.

3 Detour domination cover edge pebbling number in graphs

Theorem 3.1. For a wheel graph W_n, $n \geq 7$, $\psi_e^*(W_n) = \begin{cases} 4(n-1) + \frac{n-3}{2}, \text{ if } n \text{ is odd,} \\ 4(n-1) + \frac{n-4}{2}, \text{ if } n \text{ is even.} \end{cases}$

Proof: Case 1. n is odd.
Let W_n be the wheel graph with the edge set $E(W_n)$=$\{e_1, e_2, ..., e_{2n}\}$ i.e., together with the cycle edges $\{e_1, e_2, ..., e_n\}$ and the spokes $\{e_{n+1}, e_{n+2}, ..., e_{2n}\}$. Let e_{n+1} be the spoke adjacent to e_1 and e_n and let e_{2n} be the spoke adjacent to e_n and e_{n-1}. Then $S_1 = \{e_1, e_2, e_4, e_6, ..., e_{n-1}, e_{n+1}\}$ and $S_2 = \{e_1, e_3, e_5, e_7, ..., e_{n-2}, e_{n+2}, e_{2n}\}$ are the two extended edge-to-edge detour sets follow the different pattern of pebble distribution but both the sets will have the same number of edges for every n and also these edges form an edge dominating set. All other extended edge-to-edge detour sets follow the same pebble distribution pattern as either S_1 or S_2. So, it is enough to consider these two sets.

For $n = 7$, each extended edge-to-edge detour set contains 5 edges. For $n = 9$, each such set contains 6 edges. In general, as n increases, the cardinality of the extended edge-to-edge detour set increases by one edge.

Since the set S_2 contains more spokes than S_1, the number of pebbles required to cover all edges of S_2 is smaller than that required for S_1. Hence, it suffices to analyse the set $S_1 = \{e_1, e_2, e_4, e_6, ..., e_{n-1}, e_{n+1}\}$. Let e^* be any cycle edge of W_n. To determine the worst-case configuration, assume that all pebbles are concentrated on the single edge e^*. When $4(n\text{-}1)$pebbles are placed on e^*, all edges in the extended edge-to-edge detour set can be covered, whereas any smaller number of pebbles fails for at least one such configuration. Moreover, e^* is the only cycle edge that requires $4(n\text{-}1)$ pebbles to achieve coverage of S_1; for every other cycle edge, strictly fewer than $4(n\text{-}1)$ pebbles suffice.

Consider a distribution of $4(n\text{-}1)+\frac{n-5}{2}$ pebbles on W_n. Place $4(n\text{-}1)$-1 pebbles on the edge and then keep the edges with one pebble by placing the remaining $\frac{n-3}{2}$ number of pebbles on the cycle edges that are not in the extended edge-to-edge detour set and not

placed on the cycle edges that are second adjacent to e^* and also place the pebbles on the spokes that are adjacent to the cycle edges and make sure that these cycle edges must be adjacent to e^* and not placed the pebbles on the spokes that are adjacent to e^*. In this way of pebble distribution, we cannot cover all the edges in the set S_1.

Therefore, $\psi_e^*(W_n) \geq 4(n\text{-}1)+\frac{n-3}{2}$ & $n \geq 7$.

Let D be any distribution of $4(n\text{-}1) + \frac{n-3}{2}$ pebbles on the edges of W_n to demonstrate the sufficient part.

Sub Case 1.1. Let e_u be the source edge with maximum pebbles where $u \in C(W_n)$.

Here, if e_u has at most $4(n\text{-}1)$ pebbles and $\frac{n-3}{2}$ pebbles on the required $C(W_n)$ and $S(W_n)$, to cover all the edges in the sets S_1 and S_2 is possible. So, we can cover all the edges in the sets S_1 and S_2 with at most $4(n\text{-}1) + \frac{n-3}{2}$ pebbles.

Sub Case 1.2. Let e_v be the source edge with maximum pebbles where $v \in \mathrm{S}(W_n)$.

Here, if $P(e_u) < 4(n\text{-}1)$ pebbles and $\frac{n-1}{2}$ pebbles on the required $\mathrm{S}(W_n)$ and $\mathrm{C}(W_n)$, we can cover all the edges in the sets S_1 and S_2 since with at most 4 pebbles, to cover any cycle edges of the sets S_1 and S_2 is possible.

When the pebbles are distributed among several edges, the number of moves needed to cover the set does not exceed that required when all pebbles are concentrated on a single edge. So, by all possible means of pebble distribution, the detour domination cover edge pebbling number for the wheel graph is $\psi_e^*(W_n) = 4(n\text{-}1) + \frac{n-3}{2}$ & $n \geq 7$ when n is odd.

Case 2. n is even.

Let $S_1 = \{e_1,\ e_2,\ e_4,\ e_6,\ \ldots,\ e_{n-4},\ e_{n-2},\ e_n\}$ and $S_2 = \{e_1,\ e_3,\ e_5,\ e_7,\ \ldots,\ e_{n-3},\ e_{n-1},\ e_{n+2}\}$ are the two extended edge-to-edge detour sets follow the different pattern of pebble distribution but both the sets will have the same number of edges for every n and also these edges form an edge dominating set. All other extended edge-to-edge detour sets follow the same pebble distribution pattern as either S_1 or S_2. So, it is enough to consider these two sets.

For $n = 8$, each extended edge-to-edge detour set contains 5 edges. For $n = 10$, each such set contains 6 edges. In general, as n increases, the cardinality of the extended edge-to-edge detour set increases by one edge.

Since the set S_2 contains more spokes than S_1, the number of pebbles required to cover all edges of S_2 is smaller than that required for S_1. Hence, it suffices to analyse the set $S_1 = \{e_1,\ e_2,\ e_4,\ e_6,\ \ldots,\ e_{n-4},\ e_{n-2},\ e_n\}$. Let e^* be any cycle edge of W_n. To determine the worst-case configuration, assume that all pebbles are concentrated on the single edge e^*. When $4(n\text{-}1)$pebbles are placed on e^*, all edges in the extended edge-to-edge detour set can be covered, whereas any smaller number of pebbles fails for at least one such configuration. Moreover, e^* is the only cycle edge that requires $4(n\text{-}1)$ pebbles to achieve

coverage of S_1; for every other cycle edge, strictly fewer than $4(n\text{-}1)$ pebbles suffice.

Consider a distribution of $4(n\text{-}1)+\frac{n-6}{2}$ pebbles on W_n. Place $4(n\text{-}1)$-1 pebbles on the edge e^* and then keep the edges with one pebble by placing the remaining $\frac{n-4}{2}$ number of pebbles on the cycle edges that are not in the extended edge-to-edge detour set and not placed on the cycle edges that are second adjacent to e^* and also place the pebbles on the spokes that are adjacent to the cycle edges and make sure that these cycle edges must be adjacent to e^* and not placed the pebbles on the spokes that are adjacent to e^*. In this way of pebble distribution, we cannot cover all the edges in the set S_1.

Therefore, $\psi_e^*(W_n) \geq 4(n\text{-}1)+\frac{n-4}{2}$ & $n \geq 8$.

Let D be any distribution of $4(n\text{-}1) + \frac{n-4}{2}$ pebbles on the edges of W_n to demonstrate the sufficient part.

Sub Case 2.1. Let e_u be the source edge with maximum pebbles where $u \in C(W_n)$.

Here, if e_u has at most $4(n\text{-}1)$ pebbles and $\frac{n-4}{2}$ pebbles on the required $C(W_n)$ and $S(W_n)$, to cover all the edges in the sets S_1 and S_2 is possible. So, we can cover all the edges in the sets S_1 and S_2 with at most $4(n\text{-}1) + \frac{n-4}{2}$ pebbles.

Sub Case 2.2. Let e_v be the source edge with maximum pebbles where $v \in \mathrm{S}(W_n)$.

Here, if $P(e_u) < 4(n\text{-}1)$ pebbles and $\frac{n-2}{2}$ pebbles on the required $\mathrm{S}(W_n)$ and $\mathrm{C}(W_n)$, we can cover all the edges in the sets S_1 and S_2 since with at most 4 pebbles, to cover any cycle edges of the sets S_1 and S_2 is possible.

When the pebbles are distributed among several edges, the number of moves needed to cover the set does not exceed that required when all pebbles are concentrated on a single edge. So, by all possible means of pebble distribution, the detour domination cover edge pebbling number for the wheel graph is $\psi_e^*(W_n) = 4(n\text{-}1)+\frac{n-4}{2}$ & $n \geq 8$ when n is even.

Theorem 3.2. For a path P_n, $n \geq 6$, $\psi_e^*(P_n) = \begin{cases} 1 + 2^{n-2}+\sum_{k=1}^{\frac{n-3}{3}} 2^{n-3k}, \text{ if } n \equiv 0 \pmod 3 \\ 1 + 2^{n-2}+\sum_{k=1}^{\frac{n-4}{3}} 2^{n-3k-1}, \text{ if } n \equiv 1 \pmod 3 \\ 1 + 2^{n-2}+\sum_{k=1}^{\frac{n-5}{3}} 2^{n-3k-2}, \text{ if } n \equiv 2 \pmod 3. \end{cases}$

Proof: Case 1. $n \equiv 0 \pmod 3$.

In path graph, the extended edge-to-edge detour set together with certain additional edges, forms an edge dominating set. So, for $n \equiv 0 \pmod 3$, the set would be $\{e_1, e_4, e_7, \ldots, e_{n-2}, e_{n-1}\}$ and let it be S_1.

Take $2^{n-2}+\sum_{k=1}^{\frac{n-3}{3}} 2^{n-3k}$ pebbles for distribution. Place $2^{n-2}+\sum_{k=1}^{\frac{n-3}{3}} 2^{n-3k}$ pebbles on the edge e_1. Now, the cost to cover the edge e_{n-1} is 2^{n-2} pebbles, the cost to cover the edge e_{n-2} is 2^{n-3} pebbles, ..., the cost to cover the edge e_4 is 8 pebbles then the edge e_1

will be left uncovered.

So, $\psi_e^*(P_n) \geq 1+2^{n-2}+\sum_{k=1}^{\frac{n-3}{3}} 2^{n-3k}$; when $n \equiv 0 \pmod 3$.

Let D be any distribution of $1+2^{n-2}+\sum_{k=1}^{\frac{n-3}{3}} 2^{n-3k}$ pebbles on the edges of P_n to demonstrate the sufficient part.

Sub Case 1.1. Let either e_1 or e_{n-1} be the source edge.

Here, we can shift a pebble to the edges in the set S_1 with at most $1+2^{n-2}+\sum_{k=1}^{\frac{n-3}{3}} 2^{n-3k}$ pebbles.

Sub Case 1.2. Let e_k be the source edge where k = 2, 3, 4, ..., n-2.

Here, we can shift a pebble to the edges in the set S_1 with at most $2+\frac{2^{n-2}+\sum_{k=1}^{\frac{n-3}{3}} 2^{n-3k}}{2}$ pebbles.

When the pebbles are distributed among multiple edges, the required number of moves is no greater than in the single-edge concentration case. So, by all possible configurations, to cover all the edges in the set S_1, we need $1+ 2^{n-2}+ \sum_{k=1}^{\frac{n-3}{3}} 2^{n-3k}$ pebbles.

Thus, $\psi_e^*(P_n) = 1+2^{n-2}+\sum_{k=1}^{\frac{n-3}{3}} 2^{n-3k}$; when $n \equiv 0 \pmod 3$.

Case 2. $n \equiv 1 \pmod 3$.

For $n \equiv 1 \pmod 3$, the set would be $\{e_1, e_4, e_7, ..., e_{n-3}, e_{n-1}\}$ and let it be S_2.

Take $2^{n-2}+\sum_{k=1}^{\frac{n-4}{3}} 2^{n-3k-1}$ pebbles for distribution. Place $2^{n-2}+\sum_{k=1}^{\frac{n-4}{3}} 2^{n-3k-1}$ pebbles on the edge e_1. Now, the cost to cover the edge e_{n-1} is 2^{n-2} pebbles, the cost to cover the edge e_{n-3} is 2^{n-4} pebbles, ..., the cost to cover the edge e_4 is 8 pebbles then the edge e_1 will be left uncovered.

So, $\psi_e^*(P_n) \geq 1+2^{n-2}+\sum_{k=1}^{\frac{n-4}{3}} 2^{n-3k-1}$; when $n \equiv 1 \pmod 3$.

Let D be any distribution of $1+2^{n-2}+\sum_{k=1}^{\frac{n-4}{3}} 2^{n-3k-1}$ pebbles on the edges of P_n to demonstrate the sufficient part.

Sub Case 2.1. Let either e_1 or e_{n-1} be the source edge.

Here, we can shift a pebble to the edges in the set S_2 with at most $1+2^{n-2}+\sum_{k=1}^{\frac{n-4}{3}} 2^{n-3k-1}$ pebbles.

Sub Case 2.2. Let e_k be the source edge where k = 2, 3, 4, ..., n-2.

Here, we can shift a pebble to the edges in the set S_2 with at most $2+\frac{2^{n-2}+\sum_{k=1}^{\frac{n-4}{3}} 2^{n-3k-1}}{2}$ pebbles.

When the pebbles are distributed among several edges, the number of moves needed to cover the set does not exceed that required when all pebbles are concentrated on a single edge. So, by all possible configurations, to cover all the edges in the set S_2, we need $1+ 2^{n-2}+ \sum_{k=1}^{\frac{n-4}{3}} 2^{n-3k-1}$ pebbles.

Thus, $\psi_e^*(P_n) = 1+2^{n-2}+\sum_{k=1}^{\frac{n-4}{3}} 2^{n-3k-1}$; when $n \equiv 1 \pmod 3$.

Case 3. $n \equiv 2 \pmod 3$.

For $n \equiv 2 \pmod 3$, the set would be $\{e_1, e_4, e_7, ..., e_{n-4}, e_{n-1}\}$ and let it be S_3.

Take $2^{n-2}+\sum_{k=1}^{\frac{n-5}{3}} 2^{n-3k-2}$ pebbles for distribution. Place $2^{n-2}+\sum_{k=1}^{\frac{n-5}{3}} 2^{n-3k-2}$ pebbles either on the edge e_1 or on the edge e_{n-1}. Without loss of generality, let e_1 be the source edge. Now, the cost to cover the edge e_{n-1} is 2^{n-2} pebbles, the cost to cover the edge e_{n-4} is 2^{n-5} pebbles, ..., the cost to cover the edge e_4 is 8 pebbles then the edge e_1 will be left uncovered.

So, $\psi_e^*(P_n) \geq 1+2^{n-2}+\sum_{k=1}^{\frac{n-5}{3}} 2^{n-3k-2}$; when $n \equiv 2 \pmod 3$.

Let D be any distribution of $1+2^{n-2}+\sum_{k=1}^{\frac{n-5}{3}} 2^{n-3k-2}$ pebbles on the edges of P_n to demonstrate the sufficient part.

Sub Case 3.1. Let either e_1 or e_{n-1} be the source edge.

Here, we can shift a pebble to the edges in the set S_3 with at most $1+2^{n-2}+\sum_{k=1}^{\frac{n-5}{3}} 2^{n-3k-2}$ pebbles.

Sub Case 3.2. Let e_k be the source edge where $k = 2, 3, 4, ..., n\text{-}2$.

Here, we can shift a pebble to the edges in the set S_3 with at most $2+\frac{2^{n-2}+\sum_{k=1}^{\frac{n-5}{3}} 2^{n-3k-2}}{2}$ pebbles.

When the pebbles are distributed among multiple edges, the required number of moves is no greater than in the single-edge concentration case. So, by all possible configurations, to cover all the edges in the set S_3, we need $1+ 2^{n-2}+ \sum_{k=1}^{\frac{n-5}{3}} 2^{n-3k-2}$ pebbles.

Thus, $\psi_e^*(P_n) = 1+2^{n-2}+\sum_{k=1}^{\frac{n-5}{3}} 2^{n-3k-2}$; when $n \equiv 2 \pmod 3$.

Theorem 3.3. The detour domination cover edge pebbling number for the comb graph CG_n is $\psi_e^*(CG_n) = 2^{n+1}\text{-}3$; $n \geq 3$.

Proof: Let CG_n be the comb graph with the edge set $E(CG_n) = \{e_1, e_2, ..., e_{2n-1}\}$ i.e., together with the edges of path graph $\{e_2, e_4, e_6, ..., e_{2n-2}\}$ and the pendant edges $\{e_1, e_3, e_5, ..., e_{2n-1}\}$. In comb graph CG_n, the extended edge-to-edge detour set that also form an edge dominating set is $\{e_1, e_3, e_5, ..., e_{2n-1}\}$ and let it be S_1.

Take $2^{n+1}\text{-}4$ pebbles for distribution. Place $2^{n+1}\text{-}4$ pebbles either on the edge e_1 or e_{2n-1}. Without loss of generality, let e_1 be the source edge. Now, the cost to cover the edge e_{2n-1} is 2^n pebbles, the cost to cover the edge e_{2n-3} is 2^{n-1} pebbles, ..., the cost to cover the edge e_3 is 4 pebbles then the edge e_1 will be left uncovered.

So, $\psi_e^*(CG_n) \geq 2^{n+1}\text{-}3$; $n \geq 3$.

Let D be any distribution of $2^{n+1}\text{-}3$ pebbles on the edges of CG_n to demonstrate the sufficient part.

Case 1. Let e_u be the source edge where $u \in E(P_n)$.

Here, we can shift a pebble to the edges in the set S_1 with at most $\frac{2^{n+1}}{2}$ pebbles.

Case 2. Let e_v be the source edge where $v \in$ pendant edges of CG_n.

Here, we can shift a pebble to the edges in the set S_1 with at most $2^{n+1}\text{-}3$ pebbles.

When the pebbles are distributed among several edges, the number of moves needed to cover the set does not exceed that required when all pebbles are concentrated on a single edge. So, by all possible configurations, to cover all the edges in the set S_1, we need 2^{n+1}-3 pebbles.

Therefore, $\psi_e^*(CG_n) = 2^{n+1}$-3; $n \geq 3$.

Theorem 3.4. For a cycle C_n, $n \geq 6$,

$$\psi_e^*(C_n) = \begin{cases} 2+\sum_{k=2\&k\neq 0(mod3)}^{\frac{n}{2}-1} 2^k + 2^{\frac{n}{2}}, \text{ if } n \equiv 0, 4 \pmod 6 \\ 1+\sum_{k=1}^{\frac{n-1}{6}} 2.2^{3k}, \text{ if } n \equiv 1 \pmod 6 \\ 1+\sum_{k=1}^{\frac{n-2}{6}} 2.2^{3k} + 2^{\frac{n}{2}}, \text{ if } n \equiv 2 \pmod 6 \\ 2+\sum_{k=2\&k\neq 0(mod3)}^{\lfloor \frac{n}{2} \rfloor -1} 2^k + 2.2^{\lfloor \frac{n}{2} \rfloor}, \text{ otherwise} \end{cases}$$

Proof: Case 1. $n \equiv 0, 4 \pmod 6$

In cycle graph, the extended edge-to-edge detour set together with a few edges collectively form an edge dominating set. So, for this case, the set would be $\{e_1, e_4, e_7, e_{10}, ..., e_n\}$ and let it be S_1.

Take $1+\sum_{k=2\&k\neq 0(mod3)}^{\frac{n}{2}-1} 2^k + 2^{\frac{n}{2}}$ pebbles for distribution. Place $1+\sum_{k=2\&k\neq 0(mod3)}^{\frac{n}{2}-1} 2^k + 2^{\frac{n}{2}}$ pebbles on the edge $\frac{n}{2}$. Now, all the edges in the set S_1 will be covered with these number of pebbles except the edge which is adjacent to $\frac{n}{2}$.

So, $\psi_e^*(C_n) \geq 2+\sum_{k=2\&k\neq 0(mod3)}^{\frac{n}{2}-1} 2^k + 2^{\frac{n}{2}}$ when $n \equiv 0, 4 \pmod 6$

Let D be any distribution of $2+\sum_{k=2\&k\neq 0(mod3)}^{\frac{n}{2}-1} 2^k + 2^{\frac{n}{2}}$ pebbles on the edges of C_n to demonstrate the sufficient part.

Sub Case 1.1. Let e_u be the source edge where $u = \frac{n}{2}$ and the adjacent edges of $\frac{n}{2}$.

Here, we can shift a pebble to the edges in the set S_1 with at most $2+\sum_{k=2\&k\neq 0(mod3)}^{\frac{n}{2}-1} 2^k + 2^{\frac{n}{2}}$ pebbles.

Sub Case 1.2. Let e_v be the source edge where $v \in E(C_n)$ and $v \neq \frac{n}{2}$ and the adjacent edges of $\frac{n}{2}$.

Here, if $P(e_v) < 2+\sum_{k=2\&k\neq 0(mod3)}^{\frac{n}{2}-1} 2^k + 2^{\frac{n}{2}}$ pebbles, we can cover all the edges in the set S_1 since the number of pebbles needed to cover the edges e_1 and e_n will be reduced.

When the pebbles are distributed among several edges, the number of moves needed to cover the set does not exceed that required when all pebbles are concentrated on a single edge. So, by all possible configurations, to cover all the edges in the set S_1, we need $2 + \sum_{k=2\&k\neq 0(mod3)}^{\frac{n}{2}-1} 2^k + 2^{\frac{n}{2}}$ pebbles.

Thus, $\psi_e^*(C_n) = 2+\sum_{k=2\&k\neq 0(mod3)}^{\frac{n}{2}-1} 2^k + 2^{\frac{n}{2}}$ when $n \equiv 0, 4 \pmod 6$.

Case 2. $n \equiv 1 \pmod 6$

For this case, the set would be $\{e_1, e_4, e_7, e_{10}, ..., e_{n-3}, e_n\}$ and let it be S_2.

Take $\sum_{k=1}^{\frac{n-1}{6}} 2.2^{3k}$ pebbles for distribution. Place $\sum_{k=1}^{\frac{n-1}{6}} 2.2^{3k}$ pebbles on the edge $\lceil \frac{n}{2} \rceil$. Now, all the edges in the set S_2 will be covered with these number of pebbles except the edge $\lceil \frac{n}{2} \rceil$.

So, $\psi_e^*(C_n) \geq 1+\sum_{k=1}^{\frac{n-1}{6}} 2.2^{3k}$ when $n \equiv 1 \pmod 6$

Let D be any distribution of $1+\sum_{k=1}^{\frac{n-1}{6}} 2.2^{3k}$ pebbles on the edges of C_n to demonstrate the sufficient part.

Sub Case 2.1. Let $e_{\lceil \frac{n}{2} \rceil}$ be the source edge.

Here, we can shift a pebble to the edges in the set S_2 with at most $1+\sum_{k=1}^{\frac{n-1}{6}} 2.2^{3k}$ pebbles.

Sub Case 2.2. Let e_u be the source edge where $u \in E(C_n)$ and $u \neq \lceil \frac{n}{2} \rceil$.

Here, if $P(e_u) < 1+\sum_{k=1}^{\frac{n-1}{6}} 2.2^{3k}$ pebbles, we can cover all the edges in the set S_2 since the number of pebbles needed to cover the edges e_1 and e_n will be reduced.

When the pebbles are distributed among several edges, the number of moves needed to cover the set does not exceed that required when all pebbles are concentrated on a single edge. So, by all possible configurations, to cover all the edges in the set S_2, we need $1+\sum_{k=1}^{\frac{n-1}{6}} 2.2^{3k}$ pebbles.

Thus, $\psi_e^*(C_n) = 1+\sum_{k=1}^{\frac{n-1}{6}} 2.2^{3k}$ when $n \equiv 1 \pmod 6$

Case 3. $n \equiv 2 \pmod 6$

For this case, the set would be $\{e_1, e_4, e_7, \ldots, e_{n-1}, e_n\}$ and let it be S_3.

Take $\sum_{k=1}^{\frac{n-2}{6}} 2.2^{3k} + 2^{\frac{n}{2}}$ pebbles for distribution. Place $\sum_{k=1}^{\frac{n-2}{6}} 2.2^{3k} + 2^{\frac{n}{2}}$ pebbles on the edge $\frac{n}{2}$. Now, all the edges in the set S_3 will be covered with these number of pebbles except the edge $\frac{n}{2}$.

So, $\psi_e^*(C_n) \geq 1+\sum_{k=1}^{\frac{n-2}{6}} 2.2^{3k} + 2^{\frac{n}{2}}$ when $n \equiv 2 \pmod 6$

Let D be any distribution of $1+\sum_{k=1}^{\frac{n-2}{6}} 2.2^{3k}+2^{\frac{n}{2}}$ pebbles on the edges of C_n to demonstrate the sufficient part.

Sub Case 3.1. Let $e_{\frac{n}{2}}$ be the source edge.

Here, we can shift a pebble to the edges in the set S_3 with at most $1+\sum_{k=1}^{\frac{n-2}{6}} 2.2^{3k} + 2^{\frac{n}{2}}$ pebbles.

Sub Case 3.2. Let e_u be the source edge where $u \in E(C_n)$ and $u \neq \frac{n}{2}$.

Here, if $P(e_u) < 1+\sum_{k=1}^{\frac{n-2}{6}} 2.2^{3k} + 2^{\frac{n}{2}}$ pebbles, we can cover all the edges in the set S_3 since the number of pebbles needed to cover the edges e_1, e_n and e_{n-1} will be reduced.

When the pebbles are distributed among multiple edges, the required number of moves is no greater than in the single-edge concentration case. So, by all possible configurations, to cover all the edges in the set S_3, we need $1+\sum_{k=1}^{\frac{n-2}{6}} 2.2^{3k} + 2^{\frac{n}{2}}$ pebbles.

Thus, $\psi_e^*(C_n) = 1+\sum_{k=1}^{\frac{n-2}{6}} 2.2^{3k} + 2^{\frac{n}{2}}$ when $n \equiv 2 \pmod 6$

Case 4. $n \equiv 3, 5 \pmod 6$

For this case, the set would be $\{e_1, e_4, e_7, e_{10}, ..., e_n\}$ and let it be S_4.

Take $1+\sum_{k=2\&k\neq 0(mod3)}^{\lfloor \frac{n}{2} \rfloor -1} 2^k+2.2^{\lfloor \frac{n}{2} \rfloor}$ pebbles for distribution. Place $1+\sum_{k=2\&k\neq 0(mod3)}^{\lfloor \frac{n}{2} \rfloor -1} 2^k+2.2^{\lfloor \frac{n}{2} \rfloor}$ pebbles on the edge $\lceil \frac{n}{2} \rceil$. Now, all the edges in the set S_4 will be covered with this number of pebbles except the edge which is adjacent to $\lceil \frac{n}{2} \rceil$.

So, $\psi_e^*(C_n) \geq 2+\sum_{k=2\&k\neq 0(mod3)}^{\lfloor \frac{n}{2} \rfloor -1} 2^k + 2.2^{\lfloor \frac{n}{2} \rfloor}$ when $n \equiv 3, 5 \pmod 6$

Let D be any distribution of $2+\sum_{k=2\&k\neq 0(mod3)}^{\lfloor \frac{n}{2} \rfloor -1} 2^k + 2.2^{\lfloor \frac{n}{2} \rfloor}$ pebbles on the edges of C_n to demonstrate the sufficient part.

Sub Case 4.1. Let $e_{\lceil \frac{n}{2} \rceil}$ be the source edge.

Here, we can shift a pebble to the edges in the set S_4 with at most $2+\sum_{k=2\&k\neq 0(mod3)}^{\lfloor \frac{n}{2} \rfloor -1} 2^k+ 2.2^{\lfloor \frac{n}{2} \rfloor}$ pebbles.

Sub Case 4.2. Let e_u be the source edge where $u \in E(C_n)$ and $u \neq \lceil \frac{n}{2} \rceil$.

Here, if $P(e_u) < 2+\sum_{k=2\&k\neq 0(mod3)}^{\lfloor \frac{n}{2} \rfloor -1} 2^k + 2.2^{\lfloor \frac{n}{2} \rfloor}$ pebbles, we can cover all the edges in the set S_4 since the number of pebbles needed to cover the edges e_1 and e_n will be reduced.

When the pebbles are distributed among several edges, the number of moves needed to cover the set does not exceed that required when all pebbles are concentrated on a single edge. So, by all possible configurations, to cover all the edges in the set S_4, we need $2+\sum_{k=2\&k\neq 0(mod3)}^{\lfloor \frac{n}{2} \rfloor -1} 2^k + 2.2^{\lfloor \frac{n}{2} \rfloor}$ pebbles.

Thus, $\psi_e^*(C_n) = 2+\sum_{k=2\&k\neq 0(mod3)}^{\lfloor \frac{n}{2} \rfloor -1} 2^k + 2.2^{\lfloor \frac{n}{2} \rfloor}$ when $n \equiv 3, 5 \pmod 6$

4 Algorithm to determine the detour domination cover edge pebbling number (DDCEPN) of a graph *G*.

Input: A simple and connected graph G.

Output: $\psi_e^*(G)$, the detour domination cover edge pebbling number of a graph G.

i) Determine all the possible extended edge-to-edge detour (EEED) sets of G.

ii) For each EEED set, check whether it forms an edge dominating set. If not, add the minimum number of edges required so that the set achieves an edge dominating set.

iii) If only one such required set exists, proceed to Step 6. Or else, continue.

iv) Group the sets according to their pebble distribution patterns and eliminate the sets that follow identical patterns.

v) Retain one representative set from each distinct pebble distribution pattern.

vi) For each retained set, determine its DDCEPN i.e., minimum number of pebbles that can be placed anywhere on the edges of G, goal is to move a pebble to all edges of the set.

vii) Repeat Step 6 for all retained sets.

viii) Define $\psi_e^*(G) = \max\{\text{DDCEPN(S): } S \text{ is a retained set}\}$, since the overall requirement is controlled by the set needing the largest number of pebbles.

ix) If the graph depends on a parameter n, repeat Steps 1–8 for several values of n.
x) Generalize the examined pattern to obtain a formula for $\psi_e^*(G)$.

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