International Journal of Applied Graph Theory
Vol. 9, No. 1 (2025), 01 - 08.
$ISSN(Online): 2456-7884$

Product Eccentricity Energy of Various Graphs Using Its Eigen Values

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Abstract

Graph energy is one of the most studied areas in graph theory, this article presents the idea of product eccentricity energy (E_{PE}) and the study on properties of the characteristic polynomial obtained from the product eccentricity matrix. In addition, E_{PE} of some standard graphs are also obtained.

Key words: Product eccentricity matrix, product eccentricity eigen values, product eccentricity energy.

2020 Mathematics Subject Classification : 05C30, 05C90

1 Introduction

Graph theory is the department of discrete arithmetic, it is the thinking about the structures with their properties, objectives and their relations. It was initially helpful in solving a variety of mathematical issues, but when it was used in complex science, computer science, chemistry, and other fields, it occasionally expanded into new areas of mathematical analysis.

The graphs considered in this article are simple, loop less and connected graphs. The eccentricity of a vertex is an important idea within this framework, as it evaluates the maximum distance between 2 vertices. The distance between two vertices a and b in $V(G)$ is the shortest $a-b$ path length in G. The eccentricity of the vertex evaluates the maximum distance from a specific vertex to any other vertex in the graph. Formally, it can be expressed as:

$$\zeta(b) = max\{d(b,a) : \forall a \in V(G)\}$$

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Ψ Received on February 19, 2025 / Revised on April 26, 2025 / Accepted on April 27, 2025

The eccentricity matrix $\zeta(G)$ of a graph G is obtained from the distance matrix of G by retaining the largest distances in each row and each column and leaving zeros in the remaining ones. The eccentricity energy of G is the sum of the absolute values of the eigenvalues of $\zeta(G)$.

Let G be a graph with n vertices and m edges. Denote the absolute eigen values of G as $\lambda_i, i = 1, 2, \cdots n$ arranged in order that is not increasing as $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$. In 1978 Ivan Gutman [3] computed the energy of a graph G as $E(G) = \sum_(i = 1)^n |\lambda_i|$. Li.X, Y. Shi and I. Gutman [4] introduced the energy of graph in 2012 in which the adjacency matrix of a graph G is defined as

$$a_{ij} = \begin{cases} 1 & if \quad v_i v_j \in E \\ 0 & otherwise \end{cases}$$

Spectrum of the graph is denoted by

$$Spec(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ m_1 & m_2 & \cdots & m_n \end{pmatrix}$$

Where m_i's denote the multiplicities of the corresponding eigen value. The total of the absolute values of the adjacency matrix's eigenvalues equals the graph's energy. Later, in 2009, C. Adiga et al. [1] defined the graph's maximum degree energy, which is dependent on the related graph's maximum degree matrix. The maximum degree matrix is defined as

$$d_{ij} = \begin{cases} max\{d(v_i), d(v_j)\} & if v_i v_j \in E \\ 0 & otherwise \end{cases}$$

In 2016, Ahmed M. Naji et.al [2] defined the concept of maximum eccentricity matrix. Later, Mohammad Issa Sowaity and B.Sharada [5] in 2017 introduced the concept of sum-eccentricity energy of a graph in 2017. Inspired by this product eccentricity energy of graphs is defined and their properties are studied.

2 Product Eccentricity Energy of a Graph

Definition 2.1. The product eccentricity matrix of the graph G is denoted as $P_e(G)$ and is defined as

$$p_{ij} = \begin{cases} e(v_i).e(v_j) & if v_i v_j \in E \\ 0 & otherwise \end{cases}$$

The characteristic polynomial of the product eccentricity matrix is defined by
$|\eta I - P_e(G)|$ and the corresponding characteristic equation is $\eta I - P_e(G) = 0$. Here I is the identity matrix of order n. Eigen values of the product eccentricity matrix are the roots

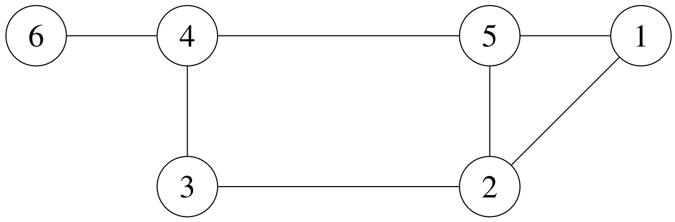
of the characteristic polynomial. $P_e(G)$ is a real symmetric matrix with its trace zero. Since G is a simple loopless graph all $a_{ii} = 0$ and its eigen values with real sum equals zero since $(tr(P_e(G) = 0)$.

$E_{PE}(G)$ is defined as the sum of the absolute eigen values,

$$E_{PE}(G) = \sum_{i=1}^{n} |\eta_i|$$

where $\eta_1, \eta_2, \cdots \eta_n$ are the eigen values of the given product eccentricity matrix.

Example 2.2. Consider the graph G_1 with 6 vertices and 7 edges



Product Eccentricity Matrix of G_1 is computed below

$$P_e(G_1) = \begin{pmatrix} 0 & 9 & 0 & 0 & 6 & 0 \\ 9 & 0 & 6 & 0 & 6 & 0 \\ 0 & 6 & 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 & 4 & 6 \\ 6 & 6 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 \end{pmatrix}$$

The Characteristic polynomial of G_1 is $\eta^6 - 257\,\eta^4 - 648\,\eta^3 + 11268\,\eta^2 + 233328\,\eta - 46656$
The eigen values are
$\eta_1 = -11.64095, \eta_2 = -8.547925, \eta_3 = -3.479582, \eta_4 = 1.28808,$
$\eta_5 = 6.650705, \eta_6 = 15.72967.$
The Product-Eccentricity energy of $G_1 = |-11.64095| + |-8.547925| + |-3.479582| + |1.28808| + |6.650705| + |15.72867| = 47.34$

3 Product Eccentricity Energy of Various Graphs and its Properties

Theorem 3.1. Let G be a graph of order n and let $c_0\eta^n + c_1\eta^{n-1} + c_2\eta^{n-2} + \cdots + c_n$ be its corresponding characteristic polynomial then,

- $c_0 = 1$
- $c_1 = 0$

- $c_2 = -\sum_{i=1,i<j}^{n}(e(v_i).e(v_j))^2$
- $c_n = (-1)^n|P_e(G)|$

Proof: 1,2. By the definition of characteristic polynomial , trivially, c_0 the coefficient of $\eta^n = 1$ and $c_1 = 0$ (Since $tr(P_e(G) = 0)$.

3. The third coefficient is obtained as

$$c_2 = \sum_{1\leq i,j\leq n}^{n} \begin{vmatrix} 0 & p_{ij} \\ p_{ij} & 0 \end{vmatrix}$$

$$= \sum_{1\leq i,j\leq n}^{n} 0 - (p_{ij})^2$$

$$= -\sum_{1\leq i,j\leq n}^{n} (p_{ij})^2$$

Here, $p_{ij} = \begin{cases} e(v_i).e(v_j) & if v_i v_j \in E \\ 0 & otherwise \end{cases}$

Thus, we have $c_2 = -\sum_{i=1,i<j}^{n} (e(v_i).e(v_j))^2$

4. For any k,

$c_k = (-1)^k \sum_{k=1}^{n} (k \times k$ principle minors)

Therefore, $c_n = (-1)^n|P_e(G)|$ ∎

Example 3.2. From the previous example co-efficient of η^4, $c_2 = -257$. By theorem 3.1 we prove

$$c_2 = -\sum_{i=1,i<j}^{n} (e(v_i).e(v_j))^2 = -[9^2 + 6^2 + 6^2 + 6^2 + 4^2 + 4^2 + 6^2]\ c_2 = 256$$

Remark 3.3. Consider a complete graph K_n, then $c_2 = \dfrac{n(n-1)}{2}$.

Corollary 3.4. For the complete graph K_n, we have

$$\sum_{i=1}^{n} (\eta_i)^2 = n(n-1)$$

Theorem 3.5. If $\eta_1, \eta_2, \cdots\cdots \eta_n$ are the product eccentricity eigen values of a graph G, then $\sum_{i=1}^{n}(\eta_i)^2 = -2c_2$

Proof: We know that the trace is the sum of the eigen values. Thus, we have

$$\sum_{i=1}^{n}(\eta_i) = trace\,(P_e(G))$$

On squaring,

$$\begin{aligned}
\sum_{i=1}^{n}(\eta_i)^2 &= (trace(P_e(G)))^2 \\
&= \sum_{i=1}^{n}\sum_{k=1}^{n} p_{ik}p_{ki} \\
&= 2\sum_{i=1}^{n}\sum_{i<k}^{n}(p_{ik})^2 \\
&= 2\sum_{i=1,i<k}^{n}(e(v_i).e(v_j))^2 \\
\sum_{i=1}^{n}(\eta_i)^2 &= -2c_2
\end{aligned}$$

■

Theorem 3.6. For the complete graph K_n, the product eccentricity eigen values are -1 and $n-1$ with multiplicities $(n-1)$ and 1 respectively and the product eccentricity energy is $2(n-1)$.

Proof: The characteristic polynomial for any complete graph is obtained as $c_0\eta^n + c_1\eta^{n-1} + c_2\eta^{n-2} + \cdots + c_n$. Calculating the product eccentricity matrix of the complete graph we have

$$P_e(K_n) = \begin{pmatrix}
0 & 1 & 1 & 1 & \cdots & \cdots & 1 \\
1 & 0 & 1 & 1 & \cdots & \cdots & 1 \\
1 & 1 & 0 & 1 & \cdots & \cdots & 1 \\
\vdots & \vdots & \vdots & \vdots & \cdots & \cdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \cdots & \cdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \cdots & \cdots & \vdots \\
1 & 1 & 1 & 1 & \cdots & \cdots & 0
\end{pmatrix}$$

$$P_e(K_n) = \begin{pmatrix} \eta & -1 & -1 & -1 & \cdots & \cdots & -1 \\ -1 & \eta & -1 & -1 & \cdots & \cdots & -1 \\ -1 & -1 & \eta & -1 & \cdots & \cdots & -1 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \cdots & \vdots \\ -1 & -1 & -1 & -1 & \cdots & \cdots & \eta \end{pmatrix}$$

$$= (\eta+1)^{n-1}(\eta-(n-1))$$

The corresponding characteristic equation is $(\eta+1)^{n-1}(\eta-(n-1)) = 0$

$$Spec(P_e(K_n) = \begin{pmatrix} -1 & n-1 \\ n-1 & 1 \end{pmatrix}$$

With observation of the matrix we draw the conclusion that the eigen values are -1 and $n-1$ with multiplicities $(n-1)$ and 1 respectively.

The Product Eccentricity Energy of $K_n = (n-1)|-1| + 1|n-1| = 2(n-1)$.
The following observations are made from the characteristic polynomial of the complete graph:

1. $c_0 = (-1)^n$
2. $c_1 = 0$
3. $c_2 = \frac{n(n-1)}{2}$
4. $c_n = n-1$

■

Example 3.7. Consider the complete graph K_4 the product eccentricity matrix is given below

$$P_e(K_4) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

The characteristic polynomial of the corresponding matrix is $\eta^4 - 6\eta^2 - 8\eta - 3$

$$Spec(P_e(K_4) = \begin{pmatrix} -1 & 3 \\ 3(times) & 1(time) \end{pmatrix}$$

The following observations are made from the above matrix,

1. $c_0 = 1$
2. $c_1 = 0$
3. $c_2 = \frac{n(n-1)}{2}$
4. $c_4 = n - 1 = 3$

The eigen values of K_4 is $-1, -1, -1, 3$. The Product eccentricity energy is 6.

Theorem 3.8. For a Cocktail Party graph $K_{n,2}$, the product eccentricity energy is $16n$.

Proof: The Product Eccentricity Matrix of the Cocktail Party Graph is obtained by the

$$p_{ij}(K_{n,2}) = \begin{cases} 4 & if v_i v_j \in E \\ 0 & otherwise \end{cases}$$

Since every vertex in the cocktail party graph has the eccentricity 2, its product eccentricity takes the value 4 if the vertices are adjacent else, it takes the value 0.

The spectrum obtained from the product eccentricity matrix of the cocktail party graph is

$$Spec(P_e(K_{n,2}) = \begin{pmatrix} 8n & -8n & 0 \\ 1 & 1 & n+2 \end{pmatrix}$$

Thus, from the spectrum we compute the eigen values $8n, -8n$.
Product Eccentricity energy is $|8n| + |-8n| = 16n$. ∎

Theorem 3.9. Let G be a complete bipartite graph $G = K_{r,s}$. Then the coefficient c_2 takes the value $-4^2 rs$, where r and s are integers with $r, s \geq 2$.

Proof: Using,

$$c_2 = -\sum_{i=1, i<j}^{n} (e(v_i).e(v_j))^2$$

Where $v_i v_j \in \mathcal{E}$, for every vertex in $K_{r,s}$ we have $e(v_i) = 2, i = 1, 2, \cdots r + s$.
Hence we have, $P_e(K_{r,s}) = \begin{cases} e(v_i).e(v_j) & if v_i v_j \in E \\ 0 & otherwise \end{cases}$ ∎

Theorem 3.10. If $G = K_{r,s}$ and $\eta_1, \eta_2, \cdots, \eta_n$ are its corresponding eigen values then

$$Spec(P_e(K_{r,s}) = \begin{pmatrix} 0 & 4\sqrt{rs} & -4\sqrt{rs} \\ n-2 & 1 & 1 \end{pmatrix}$$

Proof: The characteristic polynomial of $K_{r,s}$ is $\eta^n - 4^2 rs\eta^{n-2}$, the corresponding spectrum of the graph is given below

$$Spec(P_e(K_{r,s}) = \begin{pmatrix} 0 & 4\sqrt{rs} & -4\sqrt{rs} \\ n-2 & 1 & 1 \end{pmatrix}$$

The Product Eccentricity Energy of the Bipartite Graph is $8\sqrt{rs}$. ∎

4 Bounds of Product Eccentricity Energy

Theorem 4.1. Let G be any graph with n- vertices then we have $\sqrt{2H} \leq E_{PE}(G) \leq \sqrt{2Hn}$

Proof: Using Cauchy- Schwarz Inequality, $\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$

Take $a_i = 1$ and $b_i = |\eta_i|$ in Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} \left(\sum_{i=1}^{n} |\eta_i|\right)^2 &\leq n\left(\sum_{i=1}^{n} \eta_i^2\right) \\ (E_{PE}(G))^2 &\leq n(-2c_2) \\ (E_{PE}(G))^2 &\leq n(-2(-H)) \\ (E_{PE}(G))^2 &\leq 2nH \\ E_{PE}(G) &\leq \sqrt{2nH} \end{aligned}$$

This is an upper bound, We have $(E_{PE}(G))^2 = (\sum_{i=1}^{n} |\eta_i|^2) \leq \sum_{i=1}^{n} |\eta_i|^2 = 2H.$

Thus, we obtain $E_{PE}(G) \geq \sqrt{2H}$ which is the lower bound. Hence, the inequality holds.

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