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Upper Weak Edge Triangle Free Detour Number of a Graph

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Abstract

For any two vertices u and v in a connected graph $G = (V, E)$, the $u - v$ path P is called an $u - v$ *triangle free path* if no three vertices of P induce a triangle. The *triangle free detour distance* $D_{\triangle f}(u, v)$ is the length of a longest $u - v$ triangle free path in G. A $u - v$ path of length $D_{\triangle f}(u, v)$ is called an $u - v$ *triangle free detour.* A set $S \subseteq V$ is called a *weak edge triangle free detour set* of G if every edge of G has both ends in S or it lies on a triangle free detour joining a pair of vertices of S. The *weak edge triangle free detour number* $wdn_{\triangle f}(G)$ of G is the minimum order of its weak edge triangle free detour sets and any weak edge triangle free detour set of order $wdn_{\triangle f}(G)$ is a *weak edge triangle free detour basis* of G. A weak edge triangle free detour set S of G is called a *minimal weak edge triangle free detour set* if no proper subset of S is a weak edge triangle free detour set of G. The *upper weak edge triangle free detour number* $wdn^+_{\triangle f}(G)$ of G is the maximum order of its minimal weak edge triangle free detour set of G. We determine bounds for $wdn^+_{\triangle f}(G)$ and characterize graphs which realize these bounds. It is shown that for every pair a, b of positive integers with $2 \leq a \leq b$, there exists a connected graph G with $wdn_{\triangle f}(G) = a$ and $wdn^+_{\triangle f}(G) = b$.

Key words: triangle free detour distance, weak edge triangle free detour number, upper weak edge triangle free detour number.

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1 Introduction

By a *graph* $G = (V, E)$, we mean a finite undirected connected simple graph. For basic definitions and terminologies, we refer to Chartrand et al. [2]. The neighbourhood of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v. A vertex

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v is an *extreme vertex* if the subgraph $< N(v) >$ induced by its neighbourhood $N(v)$ is complete.

The concept of detour number was introduced by Chartrand et al. [1]. The *detour distance* $D(u, v)$ is the length of a longest $u - v$ path in G. A $u - v$ path of length $D(u, v)$ is called a $u - v$ *detour.* A set $S \subseteq V$ is called *detour set* of G if every vertex of G lies on a detour joining a pair of vertices of S. The *detour number* $dn(G)$ of G is the minimum order of its detour sets and any detour set of order $dn(G)$ is called a detour basis of G. A detour set S of G is called a *minimal detour set* if no proper subset of S is a detour set of G. The *upper detour number* $dn^+(G)$ of G is the maximum order of its minimal detour sets and any minimal detour set of order $dn^+(G)$ is an *upper detour basis* of G.

The concept of weak edge detour number was introduced and studied by Santhakumaran and Athisayanathan [4]. A set $S \subseteq V$ is called a *weak edge detour set* of G if every edge of G has both ends in S or it lies on a detour joining a pair of vertices of S. The *weak edge detour number* $dn_w(G)$ of G is the minimum order of its weak edge detour sets and any weak edge detour set of order $dn_w(G)$ is a *weak edge detour basis* of G. A weak edge detour set S of G is called a *minimal weak edge detour set* if no proper subset of S is a weak edge detour set of G. The *upper weak edge detour number* $dn_w^+(G)$ of G is the maximum order of its minimal weak edge detour set of G.

The concept of triangle free detour distance was introduced by Keerthi Asir and Athisayanathan [3]. A path P is called a *triangle free path* if no three vertices of P induce a triangle. The *triangle free detour distance* $D_{\triangle f}(u, v)$ is the length of a longest $u - v$ triangle free path in G. A $u - v$ path of length $D_{\triangle f}(u, v)$ is called a $u - v$ *triangle free detour.* The *triangle free detour eccentricity* $e_{\triangle f}(v)$ of a vertex in G is the maximum triangle free detour distance from v to a vertex of G. The *triangle free detour radius* $R_{\triangle f}$ of G is the minimum triangle free detour eccentricity among the vertices of G, while the *triangle free detour diameter* $D_{\triangle f}$ of G is the maximum triangle free detour eccentricity among the vertices of G.

The concept of triangle free detour number was introduced by Sethu Ramalingam, Keerthi Asir and Athisayanathan [5]. A set $S \subseteq V$ is called a *triangle free detour set* of G if every vertex of G lies on a triangle free detour joining a pair of vertices of S. The *triangle free detour number* $dn_{\triangle f}(G)$ of G is the minimum order of its triangle free detour sets and any triangle free detour set of order $dn_{\triangle f}(G)$ is called a *triangle free detour basis* of G. A triangle free detour set S of G is called a *minimal triangle free detour set* if no proper subset of S is a triangle free detour set of G. The *upper triangle free detour* *number* $dn^+_{\triangle f}(G)$ of G is the maximum order of its minimal triangle free detour sets and any minimal triangle free detour set of order $dn^+_{\triangle f}(G)$ is an *upper triangle free detour basis* of G.

The concept of weak edge triangle free detour number was introduced and studied by Sethu Ramalingam, Keerthi Asir and Athisayanathan [6]. A set $S \subseteq V$ is called a *weak edge triangle free detour set* of G if every edge of G has both ends in S or it lies on a triangle free detour joining a pair of vertices of S. The *weak edge triangle free detour number* $wdn_{\triangle f}(G)$ of G is the minimum order of its weak edge triangle free detour sets and any weak edge triangle free detour set of order $wdn_{\triangle f}(G)$ is a *weak edge triangle free detour basis* of G.

Throughout this paper, G denotes a connected graph with at least two vertices. The following theorems will be used in the sequel.

Theorem 1.1. [6] Every extreme-vertex of a non-trivial connected graph G belongs to every weak edge triangle free detour set of G.

Theorem 1.2. [6] Let G be a connected graph with cut-vertices and S a weak edge triangle free detour set of G. If v is a cut-vertex of G, then every component of $G - v$ contains an element of G.

Theorem 1.3. [6] If T is a tree with k end-vertices, then $wdn_{\triangle f}(T) = k$.

Theorem 1.4. [6] Let G be the complete graph K_n. Then a set $S \subseteq V$ is a weak edge triangle free detour basis of G if and only if $V \subseteq S$.

Theorem 1.5. [6] Let G be an even cycle of order $n \geq 4$. Then a set $S \subseteq V$ is a weak edge triangle free detour basis of G if and only if S consists of any two adjacent vertices or two antipodal vertices of G.

Theorem 1.6. [6] Let G be an odd cycle of order $n \geq 5$. Then a set $S \subseteq V$ is a weak edge triangle free detour basis of G if and only if S consists of any two adjacent vertices G.

Theorem 1.7. [6] Let G be a complete bipartite graph $K_{n,m}(2 \leq n \leq m)$. Then a set $S \subseteq V$ is a weak edge triangle free detour basis of G if and only if S consists of any two vertices of G.

Theorem 1.8. [6] Let G be the wheel $W_n = K_1 + C_{n-1}$ $(n \geq 6)$. Then a set $S \subseteq V$ is a weak edge triangle free detour set of G if and only if $V \subseteq S$.

Theorem 1.9. [4] For every pair a, b of positive integers with $2 \le a \le b$, there exists a connected graph G with $dn_w(G) = a$ and $dn_w^+(G) = b$.

2 Upper Weak Edge Triangle Free Detour Number

Definition 2.1. Let G be a connected graph. A weak edge triangle free detour set S of G is called a *minimal weak edge triangle free detour set* if no proper subset of S is a weak edge triangle free detour set of G. The *upper weak edge triangle free detour number* $wdn^+_{\triangle f}(G)$ of G is the maximum order of its minimal weak edge triangle free detour set of G.

Example 2.2. For the graph G given in Figure 2.1, it is easy to see that $S_1 = \{u, z\}$, $S_2 = \{v, y, z\}$, $S_3 = \{x, y, z\}$ and $S_4 = \{v, x, z\}$ are minimal weak edge triangle free detour sets and $|S_1| = 2$, $|S_2| = |S_3| = |S_4| = 3$. Clearly $wdn_{\triangle f}(G) = 2$. Let S be a weak edge triangle free detour set such that $|S_4| = 4$. Then it is easy to verify that S contain either S_1 or S_4 so that any four element weak edge triangle free detour set cannot be minimal. Also, $|V| = 5$ and it cannot be minimal. Hence $wdn^+_{\triangle f}(G) = 3$.

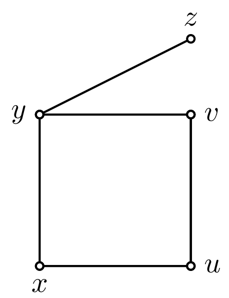

Figure 2.1 : G

Remark 2.3. Every minimum weak edge triangle free detour set is a minimal weak edge trianlge free detour set, but the converse is not true. For the graph G given in Figure 2.1, $S_2 = \{v, y, z\}$ is a minimal weak edge triangle free detour set but it is not a minimum weak edge triangle free detour set of G.

Since every minimal weak edge triangle free detour set of G is a weak edge triangle free detour set of G, we have the following theorem.

Theorem 2.4. For any connected graph G, $wdn_{\triangle f}(G) \le wdn^+_{\triangle f}(G)$.

Remark 2.5. The bound in Theorem 2.4 is sharp. For any path P_n, $wdn_{\triangle f}(G) = wdn^+_{\triangle f}(G) = 2$. Also for graph G given in Figure 2.1, $wdn_{\triangle f}(G) < wdn^+_{\triangle f}(G)$.

Now, we determine $wdn^+_{\triangle f}(G)$ for some classes of graphs.

Theorem 2.6. Let G be the complete bipartite graph $K_{n,m}(2 \leq n \leq m)$. Then a set $S \subseteq V$ is a minimal weak edge triangle free detour set of G if and only if S consists of any two vertices of G.

Proof: If S consists of any two vertices of G, then by Theorem 1.7, S is a weak edge triangle free detour basis of G so that S is a minimal weak edge triangle free detour set of G.

Conversely, assume that $S \subseteq V$ is a minimal weak edge triangle free detour set of G. If $|S| = 2$, then by Theorem 1.7, S consists of any two vertices of G. Let $|S| \geq 3$. Then by Theorem 1.7, any subset $S_1 = \{u, v\}$ of S is a weak edge triangle free detour basis of G so that S is not a minimal weak edge triangle free detour set of G, which is a contradiction. Thus S consists of any two vertices of G. ∎

Theorem 2.7. Let G be an odd cycle C_n of order $n \geq 5$. Then a set $S \subseteq V$ is a minimal weak edge triangle free detour set of G if and only if S consists of any two adjacent vertices or three independent vertices of G.

Proof: If S consists of two adjacent vertices of G, then by Theorem 1.6, S is a weak edge triangle free detour basis of G so that S is a minimal weak edge triangle free detour set of G. Let $S = \{u, v, w\}$ be such that S is an independent set. If w lies on the $u - v$ triangle free detour, then all the edges of the $u - v$ triangle free detour lie on the $u - v$ triangle free detour and the edges on the $u - v$ geodesic lie either on the $w - u$ triangle free detour or $w - v$ triangle free detour. Similarly, if w lies on the $u - v$ geodesic, then all the edges of the $u - v$ triangle free detour lie on the $u - v$ triangle free detour and the edges on the $u - v$ geodesic lie either on the $w - u$ triangle free detour or $w - v$ triangle free detour. Thus S is a weak edge triangle free detour set of G. Now, we claim that S is minimal. If $S_1 = \{x, y\}$ is any subset of S, then by Theorem 1.6, S_1 is not a weak edge triangle free detour set of G so that S is minimal weak edge triangle free detour set of G.

Conversely, assume that S is a minimal weak edge triangle free detour set of G. If $|S| = 2$, then by Theorem 1.6, S consists of two adjacent vertices of G. If $|S| = 3$, then by Theorem 1.6, the vertices of S are independent. Now, let $|S| \geq 4$. Then S must be an independent set or contains a pair of adjacent vertices of G. In either case S is not a minimal weak edge triangle free detour set of G, which is a contradiction. Thus S consists of any two adjacent vertices or three independent vertices of G. ∎

Theorem 2.8. Let G be an even cycle C_n of order $n \geq 4$. Then a set of $S \subseteq V$ is a minimal weak edge triangle free detour set of G if and only if S consists of any two adjacent vertices or two antipodal vertices or three independent vertices free from antipodal vertices of G.

Proof: If S consists of two adjacent vertices or two antipodal vertices of G, then by Theorem 1.5, S is a weak edge triangle free detour basis of G so that S is a minimal weak edge triangle free detour set of G. Let $S = \{u, v, w\}$ be such that S is an independent set free from antipodal vertices of G. Then, as in the first part of Theorem 2.7, S is a minimal weak edge triangle free detour set of G.

Conversely, assume that S is a minimal weak edge triangle free detour set of G. If $|S| = 2$, then by Theorem 1.5, S consists of two adjacent vertices or two antipodal vertices of G. If $|S| = 3$, then by Theorem 1.5, S is an independent set free from antipodal vertices of G. Now let $|S| \geq 4$. Then S must be an independent set free from antipodal vertices of G or S contains a pair of antipodal vertices of G. In any case S is not a minimal weak edge triangle free detour set of G, which is a contradiction. Thus S consists of any two adjacent vertices or two antipodal vertices or three vertices free from antipodal vertices of G. ∎

Theorem 2.9. Let G be the wheel $W_n = K_1 + C_{n-1}$ $(n \geq 6)$. Then a set $S \subseteq V$ is a minimal weak edge triangle free detour set of G if and only if $V \subseteq S$.

Proof: It is similar to that of Theorem 1.8. ∎

Corollary 2.10. Let G be a connected graph of order n.
(a) If G is the complete graph K_n, then $wdn^+_{\triangle f}(G) = n$.
(b) If G is the tree with k end-vertices, then $wdn^+_{\triangle f}(G) = k$.
(c) If G is the wheel $W_n (n \geq 6)$, then $wdn^+_{\triangle f}(G) = n$.
(d) If G is the complete bipartite graph $K_{n,m} (2 \leq n \leq m)$, then $wdn^+_{\triangle f}(G) = 2$.
(e) If G is the cycle $C_n (n > 5)$, then $wdn^+_{\triangle f}(G) = 3$.

Proof: (a) This follows from Theorem 1.1.
(b) This follows from Theorem 1.3.
(c) This follows from Theorem 1.8.
(d) This follows from Theorem 2.6.
(e) This follows from Theorems 2.7 and 2.8.

∎

The following theorem give a realization result.

Theorem 2.11. For every pair a, b of positive integers with $2 \leq a \leq b$, there exists a connected graph G with $wdn_{\triangle f}(G) = a$ and $wdn^{+}_{\triangle f}(G) = b$.

Proof: The proof is similar to that of Theorem 1.9, by replacing weak edge detour as weak edge triangle free detour and the parameter $dn_w(G)$ as $wdn_{\triangle f}(G)$. Also replacing the parameter $dn^{+}_{w}(G)$ and $wdn^{+}_{\triangle f}(G)$. ∎

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