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Total Hub Number of a Fuzzy Graph

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Abstract

In this paper we introduced the concepts of Total Hub number in fuzzy graph and is denoted by $h_t(G)$. We determine the Total Hub number $h_t(G)$ for several classes of fuzzy graph and obtain Nordhaus-Gaddum type results for this parameter. Further, some bounds of $h_t(G)$ are investigated. Also the relations between $h_t(G)$ and other known parameters in fuzzy graphs are investigated.

Key words: Graphs, fuzzy graph and total hub number in fuzzy graph
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1 Introduction

The study of Graph theory was begun by Euler in 1736 [2]. Rosenfeld [4], introduced the notion of fuzzy graph and several fuzzy analogues of graph theoretic concepts such as path, cycles and connectedness. Mahioub M.Q shubatah and Haifa A. A communicated the concept of hub number in fuzzy graph [1]. The concepts of domination in fuzzy graphs was investigated by A. Somasundaram, S. Somasundaram [5, 6]. The hub number of a graph was introduced by Matthew Walsh in [3]. Consider the fuzzy graphs that represent transportation networks that is vertices can be taken to be locations or destination and an edge exists between two vertices precisely when there is an ”easy passage” between the corresponding locations for example, a city’s network of streets, with vertices representing intersection or other points of intersect and edges road segments we are connected with a certain kind of connectivity specifically we want a sets such that any traffic between disparate points in our network passes solely through vertices in this street. In this paper we introduced the concepts of hub number of fuzzy graphs.

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2 Definitions

In this section, we review briefly some definitions in Graphs, fuzzy graphs, hub number in graph, hub number in fuzzy graph and domination number in a fuzzy graph.

A crisp graph G is a finite nonempty set of objects called vertices together with a set of unordered pairs of distinct vertices of G called edges. The vertex sets and the edges set of G are denoted by $V(G)$ and $E(G)$ respectively. Suppose that $S \subseteq V(G)$ and let $x, y \in V$. An $S-path$ between x and y is a path where all intermediate vertices are from S. A set $S \subseteq V(G)$ is called the hub set of G if it has the property that, for any $x, y \in V - S$, their is an $S - path$ in G between x and y. The smallest size of a hub set in G is called a hub number of G and is denoted by $h(G)$. Let $G = (V, \mu, \rho)$ be a fuzzy graph a vertex subset D of $V(G)$ is said to be a fuzzy Hub set of G if for any pair of vertices outside of D their is a fuzzy path between them with all intermediate vertices in D. The minimum fuzzy cardinality among all minimal fuzzy hub sets in a fuzzy graph $G$is called the hub number of G and is denoted by $h(G)$. A graph $G = (V, E)$ is called connected if each pair of vertices in G belong to a path; otherwise, G is not connected. A fuzzy graph $G = (V, \mu)$ is a set V with two function $\mu : V \to [0, 1]$ and ρ : E $\to$ [0,1] such that $\rho(\{u, v\}) \leq \mu(u) \wedge \mu(v)$ for all $u, v \in V$. We write $\rho(\{u, v\})$ for $\rho(u, v)$. The order p and size q of a fuzzy graph $G = (\mu, \rho)$ are defined to be $p = \sum_{u \in V} \mu(u)$ and $q = \sum_{(u,v) \in E} \rho(u, v)$. A path P in a fuzzy graph $G = (\mu, \rho)$ is a sequence of distinct vertices $v_0, v_1, v_2, ..., v_n$ (except possibly v_0 and v_n) such that $\mu(v_i) > o, \rho(v_{i-1},\ v_i) > 0,\ 0 \leq i \leq 1$. Here $n \geq 1$ is called the length of the path P. We say a path P is a cycle if $v_0 = v_n$ and $n \geq 3$. A fuzzy graph $G = (\mu, \rho)$ is a star if and only if $G^* = (\mu^*, \rho^*)$ is a star and is denoted by $K_{\mu(u),\ \mu(v_i)}$, where $V = u, v_i :\ 1 \leq i \leq n - 1$. A fuzzy graph $G = (\mu, \rho)$ is a wheel if and only if $G^* = (\mu^*, \rho^*)$ is a wheel and is denoted by $W_{\mu(u),\ \mu(v_i)}$. A fuzzy graph $G = (V, \mu)$ is called complete fuzzy graph if $\rho(u, v) = \mu(u) \wedge \mu(v)$ for all $u, v \in V$. A fuzzy graph G is said to be bipartite fuzzy graph if the vertex set V can be partitioned in to two nonempty sets V_1 and V_2 such that $\rho(u, v) = 0$ if $u, v \in V_1$ or $u, v \in V_2$. We say that a bipartite fuzzy graph is complete bipartite fuzzy graph if $\rho(\{u, v\}) = \mu(u) \wedge \mu(v)$ for all $u \in V_1, v\ \in V_2$.

Let $G = (\mu, \rho)$ be a fuzzy graph. Then the degree of vertex $v \in V(G)$ is defined as $d(v) = \sum_{u \neq v} \rho(u, v)$. The maximum degree of G is $\Delta(G) = \vee\{d(v) : v \in V\}$, and the minimum degree of G is $\delta(G) = \wedge\{d(v) : v \in V\}$. Let $G = (\mu, \rho)$ be a fuzzy graph and let $v \in V(G)$. Then $N(v) = \{u \in V : \rho(u, v) = \mu(u) \wedge \mu(v)\}$ is called the open neighborhood set of v and $N[v] = N(v) \cup \{v\}$ is called the closed neighborhood set of v. Let $G = (\mu, \rho)$ be a fuzzy graph and $v \in V(G)$. Then the neighborhood degree of v is defined as $d_N(v) = \sum \mu(u) : u \in N(v)$. The minimum neighborhood degree of a fuzzy graph G is $\delta_N(G) = min\{d_N(v) : v \in V\}$ and the maximum neighborhood degree of G is $\Delta_N(G) = max\{d_N(v) : v \in V\}$. The effective degree of a vertex v in fuzzy graph is denoted by $d_E(V) = \sum_1^n \rho(u, v)$. The maximum degree taken of all effective degree is denoted by $\Delta_E(G) = max\{d_E(v) : v \in V(G)\}$. The minimum effective degree of G is denoted by $\delta_E(G) = min\{d_E(v) : v \in V\}$.

A subset D of V is called the dominating set of G if for every $v \in V - D$ there exists $u \in D$ such that u dominates v. A dominating set D of a fuzzy graph G is called minimal dominating set if $D - \{v\}$ is not dominating set of G for all $v \in D$. The minimum fuzzy cardinality taken over all minimal dominating set in fuzzy graph G is called domination number of G and denoted by $\gamma(G)$. A fuzzy graph $G = (V, \mu, \rho)$ is called connected fuzzy graph if each pair of vertices in G belong to a path, otherwise, G is not connected. Let D is a $\gamma - set$ then D is connected dominating set if the fuzzy subgraph $< D >$ induced by D is connected. The connected domination number of a fuzzy graph G is the minimum cardinality taken over all connected dominating set of $G$and is denoted by $\gamma_c(G)$.

3 Main Results

The aim of this section is to introduce and study the concepts of total hub number of a fuzzy graphs.

Definition 3.1. Let $G = (V, \mu, \rho)$ be a fuzzy graph. The total hub set D of G is a vertex subset of $V(G)$ such that every pair of vertices (whether adjacent or non adjacent) of $V(G)$ are connected by a path, whose all intermediate vertices are in D.

Definition 3.2. A fuzzy total hub set D of a fuzzy graph G is called minimal total hub set if $D - \{v\}$ is not total hub set of G for all $v \in D$.

Definition 3.3. The total hub number is then defined to be the minimum fuzzy cardinality among all minimal fuzzy total hub sets in a fuzzy graph G and is denoted by $h_t(G)$.

Example 3.4. Let G be a fuzzy graph given in the Figure 3.1, where $V = \{v_1, v_2, v_3, v_4\}$, $\mu(v_1) = 0.2,\ \mu(v_2) = 0.4,\ \mu(v_3) = 0.5,\ \mu(v_4) = 0.6$ and $\rho(u, v) = \mu(u) \wedge \mu(v), \forall (u, v)\ \in \rho*$. The total hub set $S = \{v_1, v_3\}$ and hence $h_t(G) = 0.7$.

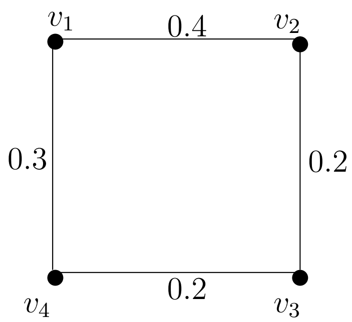

Figure: 3.1

We see that $h_t(G) = 0.3$

In the following results we give $h_t(G)$ for some standard fuzzy graphs we begin with the complete fuzzy graph K_μ.

Theorem 3.5. If $G = (V, \mu, \rho)$ is a complete fuzzy graph. Then $h_t(G) = min\{\mu(v) : v \in V(K_\mu)\}$.

Proof: Let $G = K_\mu$ be a complete fuzzy graph then for every edge $(u, v) \in \rho*, \rho(u, v) = \mu(u) \wedge \mu(v)$. That is every vertex is adjacent to the other vertices in G and all edges are effective. Thus let $S = \{u\}, u$ any vertex in G such that u has the smallest membership value in G. Then for any two vertices $x, y \in V(G)$ there is an $S - path$ between x and y. That is the vertex u is an intermediate vertices of all path joining x and y. Hence, $h_t(G) \leq |S| = min\{\mu(v) : v \in K_\mu\}$. ∎

Example 3.6. Let $G = (V, \mu, \rho)$ be a complete fuzzy graph given in Figure 3.2, where $V = \{v_1, v_2, v_3, v_4\}$, $\mu(v_1) = 0.4,\ \mu(v_2) = 0.3,\ \mu(v_3) = 0.5,\ \mu(v_4) = 0.6$ and $\rho(u, v) = \mu(u) \wedge \mu(v), u, v \in V$.

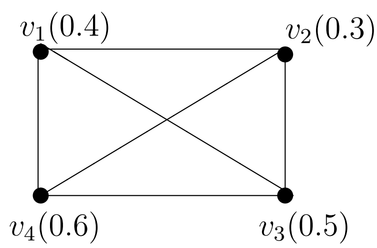

Figure: 3.2

We see that $h_t(G) = 0.3$

Proposition 3.7. For any complete fuzzy graph, $h_t(G) = \gamma(G)$.

Proof: To prove that it is enough to prove (i) $h_t(G) \leq \gamma(G)$ and (ii) $\gamma(G) \leq h_t(G)$. Now we prove (i), let G is a complete fuzzy graph and let S be a dominating set of G, since G is complete. Then for all $u, v \in V(G)$ their is an $S - path$ between them. Therefore, S is a total hub set. Hence, $h_t(G) \leq \gamma(G)$. Now we prove (ii). Let G is a complete fuzzy graph and let S is an $S - path$. Then by theorem 3.5, S contains only one vertex say x. Then for every pair of vertices $u, v \in V(G)$, x is an intermediate in any path between u and v. Then S is a dominating set. Hence, $\gamma(G) \leq |S| = h_t(G)$. Thus $h_t(G) = \gamma(G)$. ∎

The following theorem gives $h_t(G)$ of the complete bipartite fuzzy graphs $K_{\mu_1,\, \mu_2}$.

Theorem 3.8. If $G = K_{\mu_1,\, \mu_2}$ is complete bipartite fuzzy graph. Then

$$h_t(K_{\mu_1,\mu_2}) = \begin{cases} min\{\mu(v) : v \in V_2\} & if\ x, y \in V_1 \\ min\{\mu(u) : u \in V_2\} & if\, x, y \in V_2 \\ min\{\mu(u) : u \in V_1\} + min\{\mu(v) : v \in V_2\} & if\, x \in V_1,\, y \in V_2. \end{cases}$$

Proof: Let G be a complete bipartite fuzzy graph. Then we have three cases:

Case 1. If $x, y \in V_1$. Then their exits a path between x and y containing only one vertex from V_2. Hence, $h_t(K_{\mu_1,\, \mu_2}) = \min\{\mu(v) :\ v \in\ V_2\}$.

Case 2. If $x, y \in V_2$. Then their exits a path between x and y containing only one vertex from V_1. Hence, $h_t(K_{\mu_1,\, \mu_2}) = \min\{\mu(u) :\ u \in V_1\}$.

Case 3. If $x \in V_1,\ y\ \in V_2$. Then their exits path between x and y containing a vertex of V_1 and vertex of V_2. Hence $h_t(K_{\mu_1,\, \mu_2}) = min\{\mu(u) :\ u \in V_1\} + min\{\mu(v) : v\ \in\ V_2\}\ if\ \ x \in V_1,\ y \in V_2$. ∎

Theorem 3.9. Every total hub set in fuzzy graph is a hub set but the converse need not be true.

Proof: Let G be a fuzzy graph and S be a total hub set. Then for every two vertices belong $V(G)$ there is an $S - path$ between them. Thus S is a hub set. ∎

To show that the converse of the above theorem need not be true, we give the following example.

Example 3.10. Let G = (V,μ, ρ) be a fuzzy graph given in the Figure 3.3, where $V = \{v_1, v_2, v_3, v_4\}$, $\mu(v_1) = \mu(v_2) = 0.6$, $\mu(v_3) = 0.2$, $\mu(v_4) = 0.3$ and $\rho(u, v) = \mu(u) \wedge \mu(v) :\ (u, v)\ \in \rho *$.

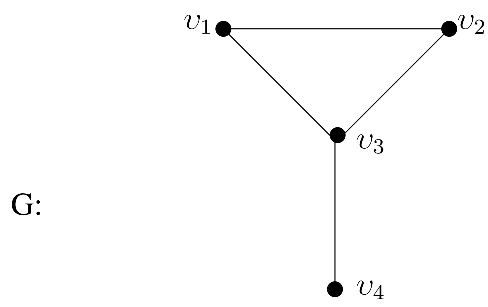

Figure: 3.3

We see that $S = \{v_3\}$ is hub set but is not total hub set.

Theorem 3.11. If G is any fuzzy graph. Then (i) $h_t(G) < p$; (ii) $h_t(\bar{G}) < p$.

Proof: Proof follows directly from the definition. ∎

Theorem 3.12. For any fuzzy graph G, $h_t(G) + h_t(\bar{G}) < 2p$.

Proof: Since $h_t(G) < p$ and $h_t(\bar{G}) < p$ by theorem 3.9. Thus $h_t(G) + h_t(\bar{G}) < 2p$. ∎

Proposition 3.13. If $G = (V, \mu, \rho)$ is a cycle fuzzy graph. Then $(i) h(G) = min \sum_{i=1}^{n-2} \mu(v_i)$ $(ii) h_t(G) = min \sum_{i=1}^{n-2} \mu(v_i)$

Theorem 3.14. For any cycle fuzzy graph $G = (V, \mu, \rho)$. Then $h(G) = h_t(G)$.

Proof: It follows by proposition 3.13. ∎

Theorem 3.15. Let $G = (V, \mu, \rho)$ be any cycle fuzzy graph such that $n \geq 5$. Then $\gamma(G) \leq h_t(G)$.

Proof: Let G be a cycle fuzzy graph and let S be a total hub set. Then S contains $n - 2$ vertices. Then for every pair of vertices $u, v \in V(G)$ their is an $S - path$ between them. Thus S is dominating set. Hence, $\gamma(G) \leq h(G) \leq h_t(G)$. ∎

Theorem 3.16. For any star fuzzy graph G, $h_t(G) = \{\mu(u) : u \; is \; a \; root\}$.

Proof: Let G be any star fuzzy graph and S = $\{u$; u is a root$\}$. Since u is root. Then u is an intermediate vertices in every path joining all two vertices in $V(G)$. Then S is an $S - path$ set. Hence, $h_t(G) = \{\mu(u) : u \; is \; a \; root\}$. ∎

Definition 3.17. A total hub set D of a fuzzy graph $G = (V, \mu, \rho)$ is called connected total hub set of G if the fuzzy subgraph $< D >$ induced by D is connected.

Definition 3.18. The minimum fuzzy cardinality taken over all connected total hub sets in G is called connected total hub number of G and is denoted by $h_{ct}(G)$.

Theorem 3.19. For any connected fuzzy graph G, $\gamma_c(G) \leq h_{ct}(G)$.

Proof: Let G be a connected fuzzy graph. Then $\gamma_c(G) \leq h_{ct}(G)$ and let S be a connected total hub set of G, since G is connected. Then for all two vertices in $V(G)$ their is an $S-path$ between them. Therefore S is a connected dominating set. Hence, $\gamma_c(G) \leq h_{ct}(G)$. ■

Theorem 3.20. For any connected fuzzy graph G, $\gamma_c(G) \leq h_t(G) + 1$.

Proof: Let G be a connected fuzzy graph. Then $\gamma_c(G) \leq h(G)+1[1]$ and since $h(G) \leq h_t(G)$. Then $h(G) + 1 \leq h_t(G) + 1$. Hence, $\gamma_c(G) \leq h(G) + 1 \leq h_t(G) + 1$. ■

Theorem 3.21. Let G be a complete fuzzy graph. Then $\triangle(G) = p - t$, $t = min\{\mu(v) : v \in V(G)\}$ if and only if $h_t(G) = t$

Proof: Let G be a complete fuzzy graph such that, $\triangle(G) = p - t$, $t = min\{\mu(v) : v \in V(G)\}$. Suppose v is a vertex of G with $deg(v) = p-t$, since G is complete fuzzy graph. Then every pair of vertices of V are connected by a path whose intermediate vertex is v. Therefore $S = \{v\}$ is a total hub set of G. Hence, $h_t(G) = t$. Conversely, suppose $h_t(G) = t$ and $S = \{v\}$ be a minimum hub set of G, since G is complete fuzzy graph. Then every total hub set is a dominating set, it follows that v is dominates all other vertices in G. Since v has the minimum fuzzy cardinality. Then $\triangle(G) = p - t$. ■

Theorem 3.22. For any fuzzy graph G. Further, equality hold if G is complete. (i) $h_t(G) \leq p - \triangle_N(G)$ (ii) $h_t(\bar{G}) < p - \triangle_N(\bar{G})$.

Proof: Let G be any fuzzy graph and let $v \in V(G)$ is a vertex such that $d_N(G) = \Delta_N(G)$. Then $V - N(v)$ is a total hub set of G. Hence, $h_t(G) \leq |V - N(v)| = p - \triangle_N(G)$. If G is a complete fuzzy graph. Then $|V - N(v)| = |S|, S$ is a total hub set contains only one vertex. Therefore, $h_t(G) = p - \triangle_N(G)$. Since $h_t(\bar{G}) < h_t(G)$. Then (ii) hold. ■

Theorem 3.23. For any fuzzy graph $G = (V, \mu, \rho)$, $h_t(G) \leq p - \delta_N$.

Proof: Let $v \in V$ with $d_N(v) = \delta_N$. Suppose S be a total hub set of G. Clearly $v \in S$ and $S \subseteq V - N(v)$. Therefore, $h_t(G) = |S| \leq |V - N(v)| = p - |N(v)| \leq p - \delta_N$. ■

Corollary 3.24. For any fuzzy graph G, $h_t(G) \leq p - \delta_E$.

Proof: Clearly $\Delta_E \leq \Delta_N$, $\delta_E \leq \delta_N$ and by the above theorem. Then $h_t(G) \leq p - \delta_E$. ■

Theorem 3.25. Let G be a fuzzy graph without isolated vertices. Then $h_t(G) \leq \frac{2p}{\Delta(G)+1}$.

Proof: Let G be a fuzzy graph have no isolated vertices and let D be a total hub set of G. Further, let $t = \sum \rho(e)$, where e is an edge in G which having one vertex in D and the other in $V - D$. Since $\Delta(G) \geq deg(v),\ v \ \in D$, and each vertex in D has at least one neighbor in D, we have $t \geq (\Delta(G) + 1)|D| = (\triangle(G) + 1)h_t(G)$. Also, since each vertex in $V - D$ is adjacent to at least two vertices in D, we have $t \leq 2|V - D| = 2(p - h_t(G))$. Hence, $2(p - h_t(G)) \geq (\triangle(G) + 1)h_t(G)$ which reduces the bound of the theorem. ■

Theorem 3.26. For any connected fuzzy graph G and $\bar{G}$, $h_t(G) + h_t(\bar{G}) < p + t,\ t = max\{\mu(v) :\ v \in V(G)\}$.

Proof: From theorem 3.14, $h_t(G) < p - \triangle(G)$ and $h_t(\bar{G}) < p - \triangle(\bar{G})$.
We have $h_t(G) + h_t(\bar{G}) < p - \triangle(G) + p - \triangle(\bar{G})$
= $2p - (\triangle(G) + \triangle(\bar{G}))$
= $2p - (\triangle(G) + p - t - \delta(G))$
= $p - t - ((\triangle(G) + \delta(G))$.
Since $((\triangle(G)+\delta(G)) \geq 0$, we have $h_t(G)+h_t(\bar{G}) < p+t,\ t = max\{\mu(v) :\ v \in V(G)\}$. ■

4 Conclusion

In this paper, the total hub number in fuzzy graphs is defined and also applied for the various types of fuzzy graphs and suitable examples are given. We have done some results with examples and relations of total hub number. Also, known parameters in fuzzy graph have been discussed with suitable examples.

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