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Seidel Laplacian Energy of Graphs

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Abstract

The Seidel matrix of the graph G of order n and of size m is defined as $S(G) = (s_{ij})$, where $s_{ij} = -1$ if the vertices v_i and v_j are adjacent, $s_{ij} = 1$ if the vertices v_i and v_j are not adjacent, and $s_{ij} = 0$ if $i = j$. Let $D_S(G) = diag(n-1-2d_1, n-1-2d_2, \ldots, n-1-2d_n)$ be a diagonal matrix in which d_i is the degree of the vertex v_i. The Seidel Laplacian matrix of G is defined as $S_L(G) = D_S(G) - S(G)$. If $\sigma_1^L, \sigma_2^L, \ldots, \sigma_n^L$ are the eigenvalues of $S_L(G)$, then the Seidel Laplacian energy of G is $\sum\limits_{i=1}^{n} \left|\sigma_i^L - \frac{n(n-1)-4m}{n}\right|$. We establish the main properties of the eigenvalues of $S_L(G)$ and of Seidel Laplacian energy.

Key words: Seidel Laplacian matrix, Seidel Laplacian eigenvalues, Seidel Laplacian energy.

2010 Mathematics Subject Classification : 05C50

1 Introduction

Let G be a simple, undirected graph with n vertices and m edges. Let $v_1, v_2, \ldots, v_n$ be the vertices of G. The degree of a vertex v_i is the number of edges incident to it and is denoted by d_i. If $d_i = r$ for all $i = 1, 2, \ldots, n$, then G is said to be an r-regular graph. The *adjacency matrix* of a graph G is the square matrix $A(G) = [a_{ij}]$, in which $a_{ij} = 1$

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if v_i is adjacent to v_j and $a_{ij} = 0$ otherwise. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A(G)$ then the ordinary energy of a graph G is defined as [11]

$$E(G) = \sum_{i=1}^{n} |\lambda_i| . \tag{1}$$

More results on the ordinary graph energy can be found in the book [17].

Let $D(G) = diag(d_1, d_2, \ldots, d_n)$ be the diagonal matrix of vertex degrees. The Laplacian matrix of G is defined as $L(G) = D(G) - A(G)$. Let $\mu_1, \mu_2, \ldots, \mu_n$ be its eigenvalues. Then the Laplacian energy of G is defined as [13]

$$LE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right| . \tag{2}$$

The *Seidel matrix* of a graph G is the $n \times n$ real symmetric matrix $S(G) = (s_{ij})$, where $s_{ij} = -1$ if the vertices v_i and v_j are adjacent, $s_{ij} = 1$ if the vertices v_i and v_j are not adjacent, and $s_{ij} = 0$ if $i = j$. The eigenvalues of the Seidel matrix, labeled as $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$, are said to be the *Seidel eigenvalues* of G and their collection is the *Seidel spectrum* of G [3]. In parallel with Eq. (1), the *Seidel energy* of the graph G is defined as [14]

$$SE = SE(G) = \sum_{i=1}^{n} |\sigma_i| . \tag{3}$$

Recent results on Seidel energy are reported in [10, 19, 20, 23, 24]. It is easy to see that $S(G) = A(\overline{G}) - A(G)$, where $\overline{G}$ is the complement of the graph G.

Motivated by the numerous results obtained in the theory of Laplacian energy (see the recent papers [4–7, 16] and the references cited therein), we now put forward the concept of Seidel Laplacian matrix and then examine the Seidel Laplacian energy.

Let $D_S(G) = diag(n-1-2d_1, n-1-2d_2, \ldots, n-1-2d_n)$. Then the *Seidel Laplacian matrix* of G is defined as

$$S_L(G) = D_S(G) - S(G) .$$

Note that $D_S(G) = D(\overline{G}) - D(G)$ and $S_L(G) = L(\overline{G}) - L(G)$.

Let $\sigma_1^L, \sigma_2^L, \ldots, \sigma_n^L$, be the eigenvalues of $S_L(G)$. In analogy to Eq. (2), we define the *Seidel Laplacian energy* of G as $SLE = SLE(G) = \sum\limits_{i=1}^{n} \left| \sigma_i^L - \frac{n(n-1)-4m}{n} \right|$.

If we denote

$$\xi_i = \sigma_i^L - \frac{n(n-1)-4m}{n} \quad \text{for} \quad i = 1,2,\ldots,n, \tag{4}$$

then,

$$SLE(G) = \sum_{i=1}^{n} |\xi_i| \,. \tag{5}$$

2 Seidel Laplacian eigenvalues

It is well known that the adjacency and Laplacian eigenvalues satisfies the relations

$$\sum_{i=1}^{n} \lambda_i = 0 \qquad \text{and} \qquad \sum_{i=1}^{n} \lambda_i^2 = 2m,$$

$$\sum_{i=1}^{n} \mu_i = 2m \qquad \text{and} \qquad \sum_{i=1}^{n} \mu_i^2 = 2m + Z_1(G),$$

where $Z_1(G) = \sum_{i=1}^{n} d_i^2$ is the so-called *first Zagreb index*, whose mathematical properties have been studied in due detail (see [2, 12]).

Lemma 2.1. Let G be a graph with n vertices and m edges. Then the eigenvalues σ_i^L, $i = 1,2,\ldots,n$, of the Seidel Laplacian matrix satisfy the relations

$$\sum_{i=1}^{n} \sigma_i^L = n(n-1) - 4m \text{ and } \qquad \sum_{i=1}^{n} (\sigma_i^L)^2 = n^2(n-1) - 8m(n-1) + 4Z_1(G) \,.$$

Proof:

$$\begin{aligned}
\sum_{i=1}^{n} \sigma_i^L &= \text{trace}[S_L(G)] = \sum_{i=1}^{n} (n-1-2d_i) = n(n-1) - 4m. \\
\sum_{i=1}^{n} (\sigma_i^L)^2 &= \text{trace}[S_L(G)^2] = \sum_{i=1}^{n} \Big[(n-1-2d_i)^2 + (n-1)\Big] \\
&= n^2(n-1) - 8m(n-1) + 4Z_1(G) \,.
\end{aligned}$$

∎

Lemma 2.2. Let ξ_i be as defined above by Eq. (4). Then

$$\sum_{i=1}^{n} \xi_i = 0 \qquad \text{and} \qquad \sum_{i=1}^{n} \xi_i^2 = M$$

where

$$M = n(n-1) + 4Z_1(G) - \frac{16m^2}{n}\,.$$

Proof:

$$\begin{aligned}\sum_{i=1}^{n} \xi_i &= \sum_{i=1}^{n}\left(\sigma_i^L - \frac{n(n-1)-4m}{n}\right) = \sum_{i=1}^{n}\sigma_i^L - [n(n-1)-4m]\\ &= n(n-1)-4m-[n(n-1)-4m] = 0\,.\end{aligned}$$

$$\begin{aligned}\sum_{i=1}^{n} \xi_i^2 &= \sum_{i=1}^{n}\left(\sigma_i^L - \frac{n(n-1)-4m}{n}\right)^2\\ &= \sum_{i=1}^{n}(\sigma_i^L)^2 - 2\left(\frac{n(n-1)-4m}{n}\right)\sum_{i=1}^{n}\sigma_i^L + \sum_{i=1}^{n}\left(\frac{n(n-1)-4m}{n}\right)^2\\ &= n^2(n-1) - 8m(n-1) + 4Z_1(G) - 2\left(\frac{n(n-1)-4m}{n}\right)[n(n-1)-4m]\\ &+ \; n\left[\frac{(n(n-1)-4m)^2}{n^2}\right] = M\,.\end{aligned}$$

∎

Proposition 2.3. If σ_i^L , $i = 1, 2, \ldots, n$, are the Seidel Laplacian eigenvalues of G, then $-\sigma_i^L$, $i = 1, 2, \ldots, n$, are the Seidel Laplacian eigenvalues of $\overline{G}$.

Proof: $S_L(G) = L(\overline{G}) - L(G) = -S_L(\overline{G})$. ∎

Theorem 2.4. If $\sigma_1, \sigma_2, \ldots, \sigma_n$ are the Seidel eigenvalues of an r-regular graph G, then the Seidel Laplacian eigenvalues of G are $n-1-2r-\sigma_i$, $i = 1, 2, \ldots, n$.

Proof: If G is an r-regular graph then

$$S_L(G) = (n-1-2r)I - S(G)$$

where I is an identity matrix. ∎

Lemma 2.5.

$$\left|\sum_{i<j} \xi_i \, \xi_j\right| = \frac{1}{2} \, M \, .$$

Proof: Since $\sum_{i=1}^{n} \xi_i = 0$, we get $\sum_{i=1}^{n} \xi_i^2 = -2 \sum_{i<j} \xi_i \xi_j$. ∎

3 Seidel Laplacian energy

Results of this section are proved using the standard techniques from the theory of graph energy [8, 11, 13, 15, 18].

Theorem 3.1. Let G be a graph with n vertices and m edges. Then
$SLE(G) \leq \sqrt{n \left[n(n-1) + 4Z_1(G) - \frac{16m^2}{n} \right]}$.

Proof: The Cauchy–Schwarz inequality states that,

$$\left(\sum_{i=1}^{n} a_i \, b_i \right)^2 \leq \left(\sum_{i=1}^{n} a_i^2 \right) \left(\sum_{i=1}^{n} b_i^2 \right) .$$

Let $a_i = 1$ and $b_i = |\xi_i|$, $i = 1, 2, \ldots, n$. Then

$$\left(\sum_{i=1}^{n} |\xi_i| \right)^2 \leq n \sum_{i=1}^{n} |\xi_i|^2 = nM$$

implying

$$SLE(G)^2 \leq n \left[n(n-1) + 4Z_1(G) - \frac{16m^2}{n} \right] .$$

∎

Theorem 3.2. Let G be a graph with n vertices and m edges. Then

$$SLE(G) \geq \sqrt{2M} \, .$$

Proof: From Eq. (5) it follows

$$SLE(G)^2 = \sum_{i=1}^{n} |\xi_i|^2 + 2 \sum_{i<j} |\xi_i||\xi_j|$$

$$\geq \sum_{i=1}^{n} |\xi_i|^2 + 2\left|\sum_{i<j} \xi_i\xi_j\right| = M + M = 2M\,.$$

■

We now state four analytical inequalities that shall be needed in the subsequent considerations.

Lemma 3.3. [21] Suppose that a_i and b_i, $1 \leq i \leq n$, are positive real numbers. Then

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \leq \frac{1}{4}\left(\sqrt{\frac{M_1\,M_2}{m_1\,m_2}} + \sqrt{\frac{m_1\,m_2}{M_1\,M_2}}\right)^2 \left(\sum_{i=1}^{n} a_i\,b_i\right)^2$$

where $M_1 = \max\limits_{1\leq i\leq n}(a_i)$, $M_2 = \max\limits_{1\leq i\leq n}(b_i)$, $m_1 = \min\limits_{1\leq i\leq n}(a_i)$, and $m_2 = \min\limits_{1\leq i\leq n}(b_i)$.

Lemma 3.4. [22] Let a_i and b_i, $1 \leq i \leq n$, be non-negative real numbers. Then

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i\,b_i\right)^2 \leq \frac{n^2}{4}\,(M_1\,M_2 - m_1\,m_2)^2$$

where M_i and m_i, $i = 1,2$, are same as in Lemma 3.3.

Lemma 3.5. [9] Suppose that a_i and b_i, $1 \leq i \leq n$, are positive real numbers. Then

$$\left|n\sum_{i=1}^{n} a_i\,b_i - \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i\right| \leq \alpha(n)(A-a)(B-b)$$

where a, b, A and B are real constants, such that for each i, $1 \leq i \leq n$, the conditions $a \leq a_i \leq A$ and $b \leq b_i \leq B$ are satisfied. Further, $\alpha(n) = n\left\lfloor\frac{n}{2}\right\rfloor\left(1 - \frac{1}{n}\left\lfloor\frac{n}{2}\right\rfloor\right)$.

Lemma 3.6. [1] Let a_i and b_i, $1 \leq i \leq n$, be non-negative real numbers. Then

$$\sum_{i=1}^{n} b_i^2 + rR\sum_{i=1}^{n} a_i^2 \leq (r+R)\left(\sum_{i=1}^{n} a_i\,b_i\right)$$

where r and R are real constants, such that for each i, $1 \leq i \leq n$, the conditions $ra_i \leq b_i \leq Ra_i$ are satisfied.

Theorem 3.7. Let G be a graph with n vertices and m edges. Let ξ_i be defined as in Eq. (4). Let $\xi_{min} = \min\limits_{1\leq i\leq n} |\xi_i|$ and $\xi_{max} = \max\limits_{1\leq i\leq n} |\xi_i|$. Then

$$SLE(G) \geq \sqrt{nM - \frac{n^2}{4}(\xi_{max} - \xi_{min})^2}\,. \tag{6}$$

Proof: Apply Lemma 3.4 for $a_i = 1$ and $b_i = |\xi_i|$. This leads to

$$\begin{aligned}\sum_{i=1}^{n} 1^2 \sum_{i=1}^{n} |\xi_i|^2 - \left(\sum_{i=1}^{n} |\xi_i|\right)^2 &\leq \frac{n^2}{4}(\xi_{max} - \xi_{min})^2 \\ nM - SLE(G)^2 &\leq \frac{n^2}{4}(\xi_{max} - \xi_{min})^2\end{aligned}$$

and inequality (6) follows. ■

Theorem 3.8. Let G be a graph with n vertices and m edges and let ξ_{min} and ξ_{max} be same as in Theorem 3.7. Then

$$SLE(G) \geq \frac{4\sqrt{nM}\sqrt{\xi_{max}\,\xi_{min}}}{\xi_{max} + \xi_{min}}\,. \tag{7}$$

Proof: Using Lemma 3.3 and setting $a_i = |\xi_i|$ and $b_i = 1$, implies that

$$\begin{aligned}\sum_{i=1}^{n} |\xi_i|^2 \sum_{i=1}^{n} 1^2 &\leq \frac{1}{4}\left(\sqrt{\frac{\xi_{min}}{\xi_{max}}} + \sqrt{\frac{\xi_{max}}{\xi_{min}}}\right)^2 \left(\sum_{i=1}^{n} |\xi_i|\right)^2 \\ nM &\leq \frac{1}{4}\left(\frac{(\xi_{max} + \xi_{min})^2}{\xi_{max}\,\xi_{min}}\right) SLE(G)^2\end{aligned}$$

and inequality (7) follows. ■

Theorem 3.9. Let G, ξ_{min}, and ξ_{max} be same as in the Theorem 3.7, and $\alpha(n)$ same as in Lemma 3.5. Then

$$SLE(G) \geq \sqrt{nM - \alpha(n)\left(\xi_{max} - \xi_{min}\right)^2}\,. \tag{8}$$

Proof: Using Lemma 3.5 and setting $a_i = |\xi_i| = b_i$, $a = \xi_{min} = b$ and $A = \xi_{max} = B$ implies that

$$\begin{aligned}\left| n\sum_{i=1}^{n} |\xi_i|^2 - \left(\sum_{i=1}^{n} |\xi_i|\right)^2 \right| &\leq \alpha(n)\left(\xi_{max} - \xi_{min}\right)^2 \\ nM - SLE(G)^2 &\leq \alpha(n)\left(\xi_{max} - \xi_{min}\right)^2\end{aligned}$$

and inequality (8) follows. ■

Since $\alpha(n) \leq \frac{n^2}{4}$, according to Theorem 3.9 we have:

Corollary 3.10. $SLE(G) \geq \sqrt{nM - \frac{n^2}{4}\left(\xi_{max} - \xi_{min}\right)^2}\,.$

Theorem 3.11. Let G, ξ_{min} and ξ_{max} be same as in the Theorem 3.7 Then

$$SLE(G) \geq \frac{M + n\,\xi_{max}\,\xi_{min}}{(\xi_{max} + \xi_{min})}\,. \tag{9}$$

Proof: Apply Lemma 3.6 and set $b_i = |\xi_i|$, $a_i = 1$, $r = \xi_{min}$ and $R = \xi_{max}$, which implies

$$\begin{aligned} \sum_{i=1}^{n} |\xi_i|^2 + \xi_{max}\,\xi_{min} \sum_{i=1}^{n} 1 &\leq (\xi_{max} + \xi_{min}) \sum_{i=1}^{n} |\xi_i| \\ M + n\,\xi_{max}\,\xi_{min} &\leq (\xi_{max} + \xi_{min})\, SLE(G) \end{aligned}$$

and inequality (9) follows. ∎

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