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On Skew-Product of RandiĆ and Sum-Connectivity Energy of Digraphs

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Abstract

In this paper we introduce the concept of skew-product of Randić and sum-connectivity energy of directed graphs. We then obtain upper and lower bounds for skew-product of Randić and sum-connectivity energy of digraphs. Then we compute the skew-product of Randić and sum-connectivity energy of some graphs such as star digraph, complete bipartite digraph, the $S_m \wedge P_2$ digraph and a crown digraph.

Key words: Skew-product of Randić and sum-connectivity energy

2010 Mathematics Subject Classification : 05C50

1 Introduction

In 2010, Bo Zhou and Nenad Trinajstic [4] have introduced the sum-connectivity energy of a graph as follows. Let G be a simple graph and let $v_1, v_2, \ldots, v_n$ be its vertices. For $i = 1, 2, \ldots, n$, let d_i denote the degree of the vertex v_i. Then the sum-connectivity matrix of G is defined as $R = (R_{ij})$, where

$$R_{ij} = \begin{cases} 0, & \text{if } i = j, \\ \frac{1}{\sqrt{d_i + d_j}}, & \text{if the vertices } v_i \text{ and } v_j \text{ are adjacent,} \\ 0, & \text{if the vertices } v_i \text{ and } v_j \text{ are not adjacent.} \end{cases}$$

The sum-connectivity energy of G is defined as the sum of absolute values of the eigenvalues of the sum-connectivity matrix of G arranged in a non-increasing order.

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In the same year, Burcu Bozkurt, Dilek Güngör, Gutman and Sinan Çevik [3], have defined the Randić energy of a graph G as the sum of the absolute values of the eigenvalues of the Randić matrix (R_{ij}) where

$$R_{ij} = \begin{cases} 0, & \text{if } i = j, \\ \frac{1}{\sqrt{d_i d_j}}, & \text{if the vertices } v_i \text{ and } v_j \text{ are adjacent,} \\ 0, & \text{if the vertices } v_i \text{ and } v_j \text{ are not adjacent.} \end{cases}$$

Motivated by these works, in [7] Poojitha and Puttaswamy define product of Randić and sum-connectivity energy as follows. The product of Randić and sum-connectivity matrix of G is the $n \times n$ matrix $A_{prs} = (a_{ij})$, where

$$a_{ij} = \begin{cases} 0, & \text{if } i = j, \\ \frac{1}{\sqrt{(d_i^2 d_j + d_i d_j^2)}}, & \text{if the vertices } v_i \text{ and } v_j \text{ are adjacent,} \\ 0, & \text{if the vertices } v_i \text{ and } v_j \text{ are not adjacent.} \end{cases}$$

The product of Randić and sum-connectivity energy of G is defined as the sum of absolute values of the eigenvalues of the product of Randić and sum-connectivity matrix of G.

In 2010, Adiga, Balakrishnan and Wasin So [1] have introduced the skew energy of a digraph as follows. Let D be a digraph of order n with vertex set $V(D) = \{v_1, v_2, \ldots, v_n\}$ and arc set $\Gamma(D) \subset V(D) \times V(D)$ where $(v_i, v_i) \notin \Gamma(D)$ for all i and $(v_i, v_j) \in \Gamma(D)$ implies $(v_j, v_i) \notin \Gamma(D)$. The skew-adjacency matrix of D is the $n \times n$ matrix $S(D) = (s_{ij})$ where $s_{ij} = 1$ whenever $(v_i, v_j) \in \Gamma(D)$, $s_{ij} = -1$ whenever $(v_j, v_i) \in \Gamma(D)$ and $s_{ij} = 0$ otherwise. Hence $S(D)$ is a skew symmetric matrix of order n and all its eigenvalues are of the form $i\lambda$ where $i = \sqrt{-1}$ and λ is a real number. The skew energy of G is the sum of the absolute values of eigenvalues of $S(D)$.

Motivated by these works, we introduce the concept of skew-product of Randić and sum-connectivity energy of a digraph as follows. Let D be a digraph of order n with vertex set $V(D) = \{v_1, v_2, \ldots, v_n\}$ and arc set $\Gamma(D) \subset V(D) \times V(D)$ where $(v_i, v_i) \notin \Gamma(D)$ for all i and $(v_i, v_j) \in \Gamma(D)$ implies $(v_j, v_i) \notin \Gamma(D)$. Then the skew-product of Randić and sum-connectivity matrix of D is the $n \times n$ matrix $A_{sprs} = (a_{ij})$ where

$$a_{ij} = \begin{cases} \frac{1}{\sqrt{(d_i^2 d_j + d_i d_j^2)}}, & \text{if } (v_i, v_j) \in \Gamma(D), \\ -\frac{1}{\sqrt{(d_i^2 d_j + d_i d_j^2)}}, & \text{if } (v_j, v_i) \in \Gamma(D), \\ 0, & \text{otherwise.} \end{cases}$$

Then the skew-product of Randić and sum-connectivity energy $E_{sprs}(D)$ of D is defined as the sum of the absolute values of eigenvalues of A_{sprs}.

In section 2 of this paper we obtain the upper and lower bounds for skew-product of Randić and sum-connectivity energy of digraphs. In Section 3 we compute the skew-product of Randić and sum-connectivity energy of some directed graphs such as complete bipartite digraph, star digraph, the $S_m \wedge P_2$ digraph and a crown digraph.

2 Upper and lower bounds for skew-product of Randić and sum-connectivity energy

Theorem 2.1. Let D be a simple digraph of order n. Then

$$E_{sprs}(D) \leq \sqrt{2n \sum_{j \sim k} \frac{1}{(d_i^2 d_j + d_i d_j^2)}}. \tag{1}$$

Proof: On using the identities,

$$\sum_{j=1}^{n} (i\lambda_j)^2 = tr(A_{sprs}^2) = -\sum_{j=1}^{n} \sum_{k=1}^{n} a_{jk}^2 = -2 \sum_{j \sim k} \frac{1}{(d_i^2 d_j + d_i d_j^2)} \tag{2}$$

and Cauchy-Schwartz inequality

$$\left(\sum_{j=1}^{n} a_j b_j \right)^2 \leq \left(\sum_{j=1}^{n} a_j^2 \right) \cdot \left(\sum_{j=1}^{n} b_j^2 \right),$$

we obtain (1).

∎

Theorem 2.2. Let D be a simple digraph of order n. Then

$$E_{sprs}(D) \geq \sqrt{2\sum_{j\sim k}\frac{1}{(d_i^2 d_j + d_i d_j^2)} + n(n-1)p^{\frac{2}{n}}},\ where\ p = |detA_{sprs}| = \prod_{j=1}^{n}|\lambda_j|. \quad (3)$$

Proof: On using again (2) and a special case of the arithmetic-geometric mean inequality, we obtain (3). ∎

3 Skew-product of Randić and sum-connectivity energies of some families of graphs

We begin with some basic definitions and notations.

Definition 3.1. [5] A graph G is said to be complete if every pair of its distinct vertices are adjacent. A complete graph on n vertices is denoted by K_n.

Definition 3.2. [5] A bigraph or bipartite graph G is a graph whose vertex set $V(G)$ can be partitioned into two subsets V_1 and V_2 such that every line of G joins V_1 with V_2. (V_1, V_2) is a bipartition of G. If G contains every line joining V_1 and V_2, then G is a complete bigraph. If V_1 and V_2 have m and n points, we write $G = K_{m,n}$. A star is a complete bigraph $K_{1,n}$.

Definition 3.3. [2] The Crown graph S_n^0 for an integer $n \geq 3$ is the graph with vertex set $\{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ and edge set $\{u_i v_j; 1 \leq i, j \leq n, i \neq j\}$. S_n^0 is therefore S_n^0 coincides with complete bipartite graph $K_{n,n}$ with the horizontal edges removed.

Definition 3.4. [6] The conjunction $(S_m \wedge P_2)$ of $S_m = \overline{K}_m + K_1$ and P_2 is the graph having the vertex set $V(S_m) \times V(P_2)$ and edge set $\{(v_i, v_j)(v_k, v_l) | v_i v_k \in E(S_m)\ and\ v_j v_l \in E(P_2)\ and\ 1 \leq i, k \leq m+1, 1 \leq j, l \leq 2\}$.

Now we compute skew-product of Randić and sum-connectivity energies of some directed graphs such as complete bipartite digraph, star digraph, the $S_m \wedge P_2$ digraph and a crown digraph.

Theorem 3.5. Let the vertex set $V(D)$ and arc set $\Gamma(D)$ of a complete bipartite digraph $K_{m,n}$.

$$V(D) = \{u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n\} \text{ and } \Gamma(D) = \{(u_i, v_j) \,|\, 1 \leq i \leq m, 1 \leq j \leq n\}.$$

Then the skew-product of Randić and sum-connectivity energy of the complete bipartite digraph is $2\sqrt{\frac{1}{m+n}}$.

Proof: The skew-product of Randić and sum-connectivity matrix of complete bipartite digraph is given by

$$A_{sprs} = \begin{pmatrix} 0 & 0 & \cdots & 0 & \gamma & \gamma & \cdots & \gamma \\ 0 & 0 & \cdots & 0 & \gamma & \gamma & \cdots & \gamma \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \gamma & \gamma & \cdots & \gamma \\ -\gamma & -\gamma & \cdots & -\gamma & 0 & 0 & \cdots & 0 \\ -\gamma & -\gamma & \cdots & -\gamma & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\gamma & -\gamma & \cdots & -\gamma & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where $\gamma = \frac{1}{\sqrt{(m+n)mn}}$. Then its characteristic polynomial is

$$\begin{aligned} |\lambda I - A_{sprs}| &= \begin{vmatrix} \lambda & 0 & \cdots & 0 & -\gamma & -\gamma & \cdots & -\gamma \\ 0 & \lambda & \cdots & 0 & -\gamma & -\gamma & \cdots & -\gamma \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda & -\gamma & -\gamma & \cdots & -\gamma \\ \gamma & \gamma & \cdots & \gamma & \lambda & 0 & \cdots & 0 \\ \gamma & \gamma & \cdots & \gamma & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma & \gamma & \cdots & \gamma & 0 & 0 & \cdots & \lambda \end{vmatrix} \\ &= \begin{vmatrix} \lambda I_m & -\frac{1}{\sqrt{(m+n)mn}} J^T \\ \frac{1}{\sqrt{(m+n)mn}} J & \lambda I_n \end{vmatrix}, \end{aligned}$$

where J is an $n \times m$ matrix with all the entries are equal to 1. Hence the characteristic equation is given by

$$\begin{vmatrix} \lambda I_m & -\frac{1}{\sqrt{(m+n)mn}} J^T \\ \frac{1}{\sqrt{(m+n)mn}} J & \lambda I_n \end{vmatrix} = 0.$$

This can be written as

$$|\lambda I_m| \left| \lambda I_n - \left(\frac{1}{\sqrt{(m+n)mn}} J \right) \frac{I_m}{\lambda} \left(-\frac{1}{\sqrt{(m+n)mn}} J^T \right) \right| = 0.$$

On simplification, we obtain

$$\frac{\lambda^{m-n}}{(m+n)mn} \left| ((m+n)mn)\lambda^2 I_n + JJ^T \right| = 0,$$

which can be written as

$$\frac{\lambda^{m-n}}{(m+n)mn}P_{JJ^T}(-((m+n)mn)\lambda^2) \;=\; 0,$$

where $P_{JJ^T}(\lambda)$ is the characteristic polynomial of the matrix ${}_mJ_n$. Thus, we have

$$\frac{\lambda^{m-n}}{(m+n)mn}(((m+n)mn)\lambda^2+mn)(((m+n)mn)\lambda^2)^{n-1}=0,$$

which is same as

$$\lambda^{m+n-2}(\lambda^2+\frac{mn}{(m+n)mn})=0.$$

Hence,

$$Spec\,(D)=\begin{pmatrix} 0 & i\sqrt{\frac{mn}{(m+n)mn}} & -i\sqrt{\frac{mn}{(m+n)mn}} \\ m+n-2 & 1 & 1 \end{pmatrix}.$$

Hence the skew-product of Randić and sum-connectivity energy of the complete bipartite digraph is

$$E_{sprs}(D)=2\sqrt{\frac{1}{m+n}},$$

as desired. ∎

Theorem 3.6. Let the vertex set $V(D)$ and arc set $\Gamma(D)$ of a Star digraph S_n.

$$V(D)=\{v_1,v_2,\ldots,v_n\} \text{ and } \Gamma(D)=\{(v_1,v_j)\,|\,2\le j\le n\}.$$

Then the skew-product of Randić and sum-connectivity energy of D is $2\sqrt{\frac{1}{n}}$.

Proof: The proof follows on replacing m by 1 and n by $n-1$ in Theorem 3.5. ∎

Theorem 3.7. Let the vertex set $V(D)$ and arc set $\Gamma(D)$ of a crown digraph S_n^0.

$$V(D)=\{u_1,u_2,\ldots,u_n,v_1,v_2,\ldots,v_n\} \text{ and } \Gamma(D)=\{(u_i,v_j)\,|\,1\le i\le n, 1\le j\le n, i\ne j\}.$$

Then the skew-skew-product of Randić and sum-connectivity energy of the crown digraph is $\frac{4}{\sqrt{2(n-1)}}$.

Proof: The skew-product of Randić and sum-connectivity matrix of crown digraph is given by

$$A_{sprs} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \gamma & \cdots & \gamma \\ 0 & 0 & \cdots & 0 & \gamma & 0 & \cdots & \gamma \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \gamma & \gamma & \cdots & 0 \\ 0 & -\gamma & \cdots & -\gamma & 0 & 0 & \cdots & 0 \\ -\gamma & 0 & \cdots & -\gamma & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\gamma & -\gamma & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

where $\gamma = \frac{1}{\sqrt{2(n-1)^3}}$. Then its characteristic polynomial is

$$|\lambda I - A_{sprs}| = \begin{vmatrix} \lambda I_n & -\frac{1}{\sqrt{2(n-1)^3}} K^T \\ \frac{1}{\sqrt{2(n-1)^3}} K & \lambda I_n \end{vmatrix},$$

where K is an $n \times n$ matrix. Hence the characteristic equation is given by

$$\begin{vmatrix} \lambda I_n & -\frac{1}{\sqrt{2(n-1)^3}} K^T \\ \frac{1}{\sqrt{2(n-1)^3}} K & \lambda I_n. \end{vmatrix} = 0.$$

This is same as

$$|\lambda I_n| \left| \lambda I_n - \left(\frac{K}{\sqrt{2(n-1)^3}} \right) \frac{I_n}{\lambda} \left(-\frac{K^T}{\sqrt{2(n-1)^3}} \right) \right| = 0,$$

which can be written as

$$\frac{1}{(2(n-1)^3)^n} P_{KK^T}(-(2(n-1)^3)\lambda^2) = 0,$$

where $P_{KK^T(\lambda)}$ is the characteristic polynomial of the matrix KK^T. Thus we have

$$\frac{1}{(2(n-1)^3)^n} [(2(n-1)^3)\lambda^2 + (n-1)^2][(2(n-1)^3)\lambda^2 + 1]^{n-1} = 0,$$

which is same as

$$\left(\lambda^2 + \frac{(n-1)^2}{2(n-1)^3} \right) \left(\lambda^2 + \frac{1}{2(n-1)^3} \right)^{n-1} = 0.$$

Therefore

$$Spec\,(D) = \begin{pmatrix} i\zeta & -i\zeta & i\tau & -i\tau \\ 1 & 1 & n-1 & n-1 \end{pmatrix},$$

where $\zeta = \sqrt{\frac{(n-1)^2}{2(n-1)^3}}$ and $\tau = \frac{1}{\sqrt{2(n-1)^3}}$. Hence the skew-product of Randić and sum-connectivity energy of crown digraph is

$$E_{sprs}(D) = \frac{4}{\sqrt{2(n-1)}},$$

as desired. ∎

Theorem 3.8. Let the vertex set $V(D)$ and arc set $\Gamma(D)$ of a digraph $S_m \wedge P_2$.

$$V(D) = \{v_1, v_2, \ldots, v_{2m+2}\} \text{ and}$$
$$\Gamma(D) = \{(v_1, v_j), (v_{m+2}, v_k) \,|\, 2 \le k \le m+1, m+3 \le j \le 2m+2\}.$$

Then the skew-product of Randić and sum-connectivity energy of D is $4\sqrt{\frac{1}{n}}$.

Proof: The skew-product of Randić and sum-connectivity matrix of $S_m \wedge P_2$ digraph is given by

$$A_{sprs} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \gamma & \cdots & \gamma \\ 0 & 0 & \cdots & 0 & -\gamma & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & -\gamma & 0 & \cdots & 0 \\ 0 & \gamma & \cdots & \gamma & 0 & 0 & \cdots & 0 \\ -\gamma & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\gamma & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{2n\times 2n},$$

where $m+1 = n$ and $\gamma = \frac{1}{\sqrt{n(n-1)}}$. Then its characteristic polynomial is given by

$$|\lambda I - A_{sprs}| = \begin{vmatrix} \lambda & 0 & \cdots & 0 & 0 & -\gamma & \cdots & -\gamma \\ 0 & \lambda & \cdots & 0 & \gamma & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda & \gamma & 0 & \cdots & 0 \\ 0 & -\gamma & \cdots & -\gamma & \lambda & 0 & \cdots & 0 \\ \gamma & 0 & \cdots & 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma & 0 & \cdots & 0 & 0 & 0 & \cdots & \lambda \end{vmatrix}_{2n\times 2n}.$$

Hence the characteristic equation is given by

$$\left(\frac{1}{\sqrt{n(n-1)}}\right)^{2n}\begin{vmatrix} \Lambda & 0 & \cdots & 0 & 0 & -1 & \cdots & -1 \\ 0 & \Lambda & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Lambda & 1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & -1 & \Lambda & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 0 & \Lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & \Lambda \end{vmatrix}_{2n\times 2n} = 0,$$

$$where\ \Lambda = \sqrt{n(n-1)}\lambda.$$

Let

$$\phi_{2n}(\Lambda) = \begin{vmatrix} \Lambda & 0 & 0 & \cdots & 0 & 0 & -1 & -1 & \cdots & -1 \\ 0 & \Lambda & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \Lambda & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda & 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & -1 & \cdots & -1 & \Lambda & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & \Lambda & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \Lambda \end{vmatrix}_{2n\times 2n}.$$

$$= (-1)^{2n+2n}\Lambda\begin{vmatrix} \Lambda & 0 & 0 & \cdots & 0 & 0 & -1 & -1 & \cdots & -1 \\ 0 & \Lambda & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \Lambda & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda & 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & -1 & \cdots & -1 & \Lambda & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & \Lambda & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \Lambda \end{vmatrix}_{(2n-1)\times(2n-1)}$$

$$+(-1)^{2n+1}\begin{vmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & -1 & \cdots & -1 & -1 \\ \Lambda & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & \Lambda & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda & 1 & 0 & \cdots & 0 & 0 \\ -1 & -1 & -1 & \cdots & -1 & \Lambda & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \Lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \Lambda & 0 \end{vmatrix}_{(2n-1)\times(2n-1)}.$$

Let

$$\Psi_{2n-1}(\Lambda) = (-1)^{2n+1}\begin{vmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & -1 & \cdots & -1 & -1 \\ \Lambda & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & \Lambda & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda & 1 & 0 & \cdots & 0 & 0 \\ -1 & -1 & -1 & \cdots & -1 & \Lambda & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \Lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \Lambda & 0 \end{vmatrix}_{(2n-1)\times(2n-1)}$$

Using the properties of the determinants, we obtain, after some simplifications

$$\Psi_{2n-1}(\Lambda) = \Lambda^{n-2}\Theta_n(\Lambda),$$

where $\Theta_n(\Lambda)=\begin{vmatrix} \Lambda & 0 & 0 & \cdots & 1 \\ 0 & \Lambda & 0 & \cdots & 1 \\ 0 & 0 & \Lambda & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & \Lambda \end{vmatrix}_{n\times n}.$

Then

$$\phi_{2n}(\Lambda) = \Lambda^{n-2}\Theta_n(\Lambda) + \Lambda\phi_{2n-1}(\Lambda).$$

Now, proceeding as above, we obtain

$$\phi_{2n-1}(\Lambda) \quad = \quad (-1)^{(2n-1)+1}\Psi_{2n-2}(\Lambda) + (-1)^{(2n-1)+(2n-1)}\Lambda\phi_{2n-2}(\Lambda)$$

$$= \Lambda^{n-3}\Theta_n(\Lambda) + \Lambda\phi_{2n-2}(\Lambda).$$

Proceeding like this, we obtain at the $(n-1)^{th}$ step

$$\phi_{2n}(\Lambda) = (n-1)\Lambda^{n-2}\Theta_n(\Lambda) + \Lambda^{(n-1)}\xi_{n+1}(\Lambda),$$

where $\xi_{n+1}(\Lambda) = \begin{vmatrix} \Lambda & 0 & 0 & \cdots & 0 \\ 0 & \Lambda & 0 & \cdots & 1 \\ 0 & 0 & \Lambda & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -1 & -1 & \cdots & \Lambda \end{vmatrix}_{(n+1)\times(n+1)}$.

$$\begin{aligned} \phi_{2n}(\Lambda) &= (n-1)\Lambda^{n-2}\Theta_n(\Lambda) + \Lambda^{n-1}\Lambda\Theta_n(\Lambda) \\ &= (n-1)\Lambda^{n-2}\Theta_n(\Lambda) + \Lambda^{n}\Theta_n(\Lambda) \\ &= ((n-1)\Lambda^{n-2} + \Lambda^{n})\Theta_n(\Lambda). \end{aligned}$$

Using the properties of the determinants, we obtain

$$\Theta_n(\Lambda) = (n-1)\Lambda^{n-2} + \Lambda^n.$$

Therefore

$$\phi_{2n}(\Lambda) = ((n-1)\Lambda^{n-2} + \Lambda^n)^2.$$

Hence characteristic equation becomes

$$\left(\frac{1}{\sqrt{n(n-1)}}\right)^{2n} \phi_{2n}(\Lambda) = 0,$$

which is same as

$$\left(\frac{1}{\sqrt{n(n-1)}}\right)^{2n} ((n-1)\Lambda^{n-2} + \Lambda^n)^2 = 0.$$

This reduces to

$$\lambda^{2n-4}((n-1) + (n(n-1))\lambda^2)^2 = 0.$$

Therefore

$$Spec\,(D) = \begin{pmatrix} 0 & i\sqrt{\frac{n-1}{n(n-1)}} & -i\sqrt{\frac{n-1}{n(n-1)}} \\ 2n-4 & 2 & 2 \end{pmatrix}.$$

Hence the skew-product of Randić and sum-connectivity energy of $S_m \wedge P_2$ digraph is

$$E_{sprs}(D) = 4\sqrt{\frac{1}{n}}.$$

■

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