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Inverse Sum Indeg Energy of a Graph

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Abstract

The inverse sum indeg matrix $A_{ISI}(G)$ of a graph G is defined so that its (i,j)-entry is equal to $\frac{d_i d_j}{d_i + d_j}$ for the vertex $v_i v_j$ and 0 otherwise. We discuss some properties of the spectral radius of A_{ISI}. The inverse sum indeg energy $E_{ISI}(G)$ of a graph G are established. Upper and lower bounds of E_{ISI} are derived. Finally, we derive a relation between E_{ISI} and some topological indices.

Key words: Inverse sum indeg matrix, inverse sum indeg eigenvalues, inverse sum indeg energy of a graph.

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1 Introduction

In this paper, all graphs are assumed to be finite simple graphs. A graph $G = (V, E)$ is a simple graph, that is, having no loops, no multiple and directed edges. We denote n to be the order and m to be the size of the graph G. For a vertex $v \in V$, the open neighborhood of v in a graph G, denoted $N(v)$, is the set of all vertices that are adjacent to v and the closed neighborhood of v is $N[v] = N(v) \cup \{v\}$. The degree of a vertex v_i in G is $d_i = d(v_i) = |N(v_i)|$. A vertex of degree one is called pendant vertex. A graph G is said to be k-regular graph if $d(v) = k$ for every $v \in V(G)$. The distance $d(u, v)$ between any two vertices u and v in a graph G is the length of the shortest path connecting them. A coloring of a graph is an assignment of colors to its vertices so that no two adjacent vertices have the same color. The chromatic number χ is defined as the minimum number of colors assigned to the vertices of a graph. All the definitions and terminologies about the graph in this paragraph are available in [11].

The concept energy of a graph was introduced by Gutman [8], in (1978). Let G be a graph with n vertices and m edges and let $A(G) = (a_{ij})$ be the adjacency matrix of G, where

$$a_{ij} = \begin{cases} 1, & if\ v_iv_j \in E, \\ 0, & \text{otherwise.} \end{cases}$$

The eigenvalues $\lambda_1,\ \lambda_2, ...,\ \lambda_n$ of a matrix $A(G)$, assumed in non-increasing order, are the eigenvalues of the graph G [3]. Let $\lambda_1 >\ \lambda_2 > ... >\ \lambda_t$ for $t \leq n$ be the distinct eigenvalues of G with multiplicities $m_1,\ m_2, ...,\ m_t$, respectively, the maximum absolute value of the eigenvalues of G is called the spectral radius of the graph G, the multiset of eigenvalues of $A(G)$ is called the spectrum of G and denoted by

$$Sp(G) = \begin{bmatrix} \lambda_1 & \lambda_2 & ... & \lambda_t \\ m_1 & m_2 & ... & m_t \end{bmatrix}$$

As A is real symmetric matrix with zero trace, the eigenvalues of G are real with sum equal to zero. The energy $E(G)$ of a graph G is defined to be the sum of the absolute values of the eigenvalues of G [8], i.e.,

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$

For more details on the mathematical aspects of the theory of graph energy we refer to [1, 7, 3] and the references therein.

The eccentricity extended energy $E_{eex}(G)$ of a graph G, was defined by Sowaity et al. [20], to be the energy of the eccentricity extended matrix $A_{eex}(G)$ of a graph G. They also studied some bounds of $E_{eex}(G)$ of a graph G. For details about other energies the authors advice to see [17, 18, 19, 21].

The inverse sum indeg index $ISI(G)$ of a graph G was defined by K. Pattarbiraman [15] as the sum of the terms $\frac{d_i d_j}{d_i+d_j}$, for $v_i v_j \in E$ and 0 otherwise, which was selected as significant predictors of phisicochemical properties of total surface area of octane isomers and for other external graphs obtained with help of Mathematical Chemistry have a particularly simple elegant structure [22]. Motivated by this, we introduce the inverse sum indeg matrix $A_{ISI}(G)$ of a graph G and derive the inverse sum indeg energy $E_{ISI}(G)$ of G. For details see [6, 16].
The classical first and second Zagreb indices which were introduced by Gutman and Trinajestic [10], in 1972 and elaborated in [9]. They are defined as:

$$M_1(G) = \sum_{i=1}^{n} d_i^2 \qquad and \qquad M_2(G) = \sum_{v_i v_j \in E(G)} d_i d_j.$$

The general Randić connectivity index is defined as [2, 13]:

$$R_\alpha = R_\alpha(G) = \sum_{v_i v_j \in E} (d_i d_j)^\alpha$$

where α is real number.
The harmonic index $H(G)$ of a graph G was introduced by L. Zhong [23], and defined as:

$$H(G) = \sum_{v_i v_j \in E} \frac{2}{d_i + d_j}.$$

2 Inverse sum indeg energy of graphs

In this section, we define the inverse sum indeg matrix $A_{ISI}(G)$ of a graph G. The inverse sum indeg energy $E_{ISI}(G)$ are established, and we discuss some properties of the spectral radius of $A_{ISI}(G)$. The starting is with the definition of $A_{ISI}(G)$ which is explained in the following definition.

Let G be a graph with n vertices. Then the inverse sum indeg matrix $A_{ISI}(G)$ of G, is defined as $A_{ISI}(G) = (s_{ij})$, where

$$s_{ij} = \begin{cases} \frac{d_i d_j}{d_i + d_j}, & if\ v_i v_j \in E, \\ 0, & \text{otherwise.} \end{cases}$$

The characteristic polynomial of the inverse sum indeg matrix $A_{ISI}(G)$ is defined by

$$P(G, \zeta) = \det(\zeta I - A_{ISI}(G)),$$

where I is the identity matrix of order n. The eigenvalues of the inverse sum indeg matrix $A_{ISI}(G)$ are the roots of the characteristic polynomial.
Since $A_{ISI}(G)$ is real symmetric with zero trace, its eigenvalues must be real with sum equal to zero, i.e., $trace(A_{ISI}(G)) = 0$. We label the eigenvalues $\zeta_1,\ \zeta_2, ...,\ \zeta_n$ in a non-increasing manner $\zeta_1 \geq \zeta_2 \geq ... \geq \zeta_n$. The inverse sum indeg energy of a graph G is denoted by $E_{ISI}(G)$ and defined as the summation of the absolute value of the eigenvalues

$$E_{ISI}(G) = \sum_{i=1}^{n} |\zeta_i|.$$

The following example explain the concept.

Let G_1 be the graph as in Figure 1.

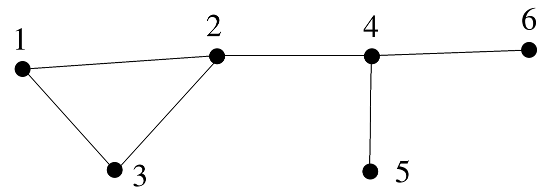

Figure 1: G_1

Then the inverse sum indeg matrix of G_1 is

$$A_{ISI}(G_1) = \begin{bmatrix} 0 & \frac{6}{5} & 1 & 0 & 0 & 0 \\ \frac{6}{5} & 0 & \frac{6}{5} & \frac{3}{2} & 0 & 0 \\ 1 & \frac{6}{5} & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & 0 & \frac{3}{4} & \frac{3}{4} \\ 0 & 0 & 0 & \frac{3}{4} & 0 & 0 \\ 0 & 0 & 0 & \frac{3}{4} & 0 & 0 \end{bmatrix}$$

The characteristic polynomial of $A_{ISI}(G_1)$ is

$$P(G_1, \zeta) = |\zeta I_n - A_{ISI}(G_1)|$$
$$= \zeta^6 - 7.255\zeta^4 + 2.88\zeta^3 + 6.615\zeta^2 + 3.24\zeta.$$

The inverse sum indeg eigenvalues of G_1 are
$\zeta_1 = 2.692$, $\zeta_2 = 1.044$, $\zeta_3 = 0$, $\zeta_4 = -0.521$, $\zeta_5 = -1$, $\zeta_6 = -2.214$.
Therefore the inverse sum indeg energy

$$E_{ISI}(G_1) = 7.472.$$

The folowing results are useful in the subsequent section

Lemma 2.1. [12] Let $B = (b_{ij})$ and $H = (h_{ij})$ be symmetric, non-negative matrices of order n. If $B \geq H$, i.e., $b_{ij} \geq h_{ij}$ for all i, j, then $\rho_1(B) \geq \rho_1(H)$, where ρ_1 is the largest eigenvalue.

Lemma 2.2. [4] Let G be a graph of order n with m edges. Then

$$\lambda_1 \geq \frac{2m}{n}$$

with equality holding if and only if G is regular graph.

Lemma 2.3. [5] If G is a graph with n vertices and chromatic number χ, then

$$\chi \geq \frac{n}{n - \lambda_1}.$$

3 Some results on inverse sum indeg spectral radius

Theorem 3.1. Let G be a r-regular graph. Then

$$A_{ISI} = \frac{r}{2}A.$$

Proof: Let G be a r-regular graph. Then

$$s_{ij} = \frac{d_i d_j}{d_i + d_j} = \frac{r^2}{2r} = \frac{1}{2}r, \text{ for all } i, j = 1, 2, ..., n.$$

Thus, the result follows. ∎

Theorem 3.2. Let $G = K_{a,b}$ be a complete bipartite graph. Then

$$A_{ISI} = \frac{ab}{a+b}A.$$

Proof: Let G be a complete bipartite graph. Then for any two adjacent vertices

$$s_{ij} = \frac{d_i d_j}{d_i + d_j} = \frac{ab}{a+b}, \text{for all } i, j = 1, 2, ..., n.$$

Thus,

$$A_{ISI} = \frac{ab}{a+b}A.$$

■

Corollary 3.3. For the regular complete bipartite graph $K_{a,a}$

$$A_{ISI} = \frac{a}{2}A.$$

Theorem 3.4. Let G be a graph with n vertices and m edges. If G has no pendent vertex, then

$$\zeta_1(G) \geq \lambda_1 \geq \frac{2m}{n},$$

with equality if and only if G is a cycle.

Proof: Let G be a graph of order n and size m. Assume that G has no pendent vertex, then $d_i \geq 2$ for all $i = 1, 2, \ldots, n$. Thus

$$d_i + d_j \leq d_i d_j \Leftrightarrow \frac{d_i d_j}{d_i + d_j} \geq 1.$$

Hence

$$A_{ISI}(G) \geq A(G).$$

Thus, by using Lemma 2.1, the result follows.
To show the equality, let $\zeta_1 = \lambda_1 = \frac{2m}{n}$. Then, by using Lemma 2.2, we get that G is regular, which comes from $\lambda_1 = \frac{2m}{n}$.
Let $\zeta_1 = \lambda_1 = \frac{2m}{n}$, then $A_{ISI} = A$, which holds if and only if $d_i = d_j = 2$, for all $v_i v_j \in E$. Hence G is a cycle.
If we assume that G is a cycle, then easily we can get the result. ■

Theorem 3.5. Let G be a star or union of stars. Then

$$\zeta_1(G) \leq \lambda_1.$$

Proof: Let G be a star or union of stars. Then for any two adjacent vertices there is at least one pendent vertex. Thus, if $v_iv_j \in E$, then $d_i = 1$ or $d_j = 1$ for all $i = 1, 2, \ldots, n$, which implies for all $v_iv_j \in E$

$$\frac{d_i d_j}{d_i + d_j} \leq 1.$$

Hence, $A_{ISI} \leq A$, and by using Lemma 2.1, we get

$$\zeta_1 \leq \lambda_1.$$

∎

4 Bounds for inverse sum indeg energy

In this section, we give some upper and lower bounds for the inverse sum indeg energy $E_{ISI}(G)$ of a graph G.

Theorem 4.1. Let G be a graph of order n and size m. Then

$$E_{ISI}(G) \leq \frac{\Delta}{2\delta}\sqrt{nm}.$$

Proof: Let G be a graph with n vertices and m edges. By using Cauchy-Schwartz inequality

$$\begin{aligned} E_{ISI} &\leq \sqrt{n\sum_{i=1}^{n}\zeta_i^2} \\ &= \sqrt{n\sum_{v_iv_j\in E}\left(\frac{d_i d_j}{d_i + d_j}\right)^2} \\ &\leq \sqrt{n\sum_{v_iv_j\in E}\frac{\Delta^2}{(2\delta)^2}} \\ &= \frac{\Delta}{2\delta}\sqrt{nm}. \end{aligned}$$

∎

Corollary 4.2. Let G be a r-regular graph. Then

$$E_{ISI} \leq \frac{n}{2}\sqrt{\frac{r}{2}}.$$

Theorem 4.3. Let G be a graph of order $n \geq 2$ and size m. If G is union of stars, then

$$E_{ISI} \leq \frac{n(\chi-1)}{\chi} + \sqrt{(n-1)\left[\frac{n\Delta^4}{4\delta^2} - \frac{n^2(\chi-1)^2}{\chi^2}\right]},$$

where χ is the chromatic number of G.

Proof: Let G be a graph of order $n \geq 2$ and size m. If G union of stars, then by using Theorem 3.5,

$$\zeta_1 \leq \lambda_1. \tag{1}$$

Now,

$$E_{ISI} = \sum_{i=1}^{n} |\zeta_i| = \zeta_1 + \sum_{i=2}^{n} |\zeta_i| \tag{2}$$

By Cauchy-Schwartz inequality

$$\sum_{i=2}^{n} \zeta_i \leq \sqrt{(n-1)\sum_{i=2}^{n} \zeta_i^2}. \tag{3}$$

But,

$$\sum_{i=2}^{n} \zeta_i^2 = \sum_{i=1}^{n} \zeta_i^2 - \zeta_1^2. \tag{4}$$

Also,

$$\begin{aligned} \sum_{i=1}^{n} \zeta_i^2 &= tr(A_{ISI}^2) \\ &= \sum_{v_iv_j \in E} \left(\frac{d_id_j}{d_i+d_j}\right)^2 \\ &\leq \frac{m\Delta^4}{4\delta^2}. \end{aligned} \tag{5}$$

By Lemma 2.3,

$$\chi \geq \frac{n}{n-\lambda_1} \Leftrightarrow \lambda_1 \leq \frac{n(\chi-1)}{\chi}. \tag{6}$$

Thus, from 1, we get

$$\zeta_1 \leq \frac{n(\chi-1)}{\chi}. \tag{7}$$

Hence, by substituting 5 in 4, 4 in 3 and 3 in 2, we get

$$E_{ISI} \leq \zeta_1 + \sqrt{(n-1)\left[\frac{n\Delta^4}{4\delta^2} - \zeta_1^2\right]}. \tag{8}$$

By substituting 7 in 8, we get the wanted result. ∎

Theorem 4.4. Let G be a nonsingular graph with n vertices and m edges. Then

$$E_{ISI}(G) \geq \zeta_1 + n - 1 + ln|det(A_{ISI})| - ln\zeta_1.$$

Proof: Since G is nonsingular graph, then $|\zeta_i| > 0$ for all $i = 1, 2, ..., n$.
If we consider the function $f(x) = x - 1 - ln\ x$. Then easy calculations give $f(x)$ is decreasing on $0 < x \leq 1$ and is increasing when $x > 1$. Also we have $f(1) = 0$, so

$$f(x) \geq 0,\ for\ x > 0.$$

Applying $f(x)$ on E_{ISI}, we have

$$\begin{aligned} E_{ISI}(G) &= \sum_{i=1}^{n} |\zeta_i| \\ &= \zeta_1 + \sum_{i=2}^{n} |\zeta_i| \\ &\geq \zeta_1 + \sum_{i=2}^{n} (1 + ln\ |\zeta_i|) \\ &= \zeta_1 + n - 1 + \sum_{i=2}^{n} ln\ |\zeta_i| \\ &= \zeta_1 + n - 1 + ln|\prod_{i=2}^{n} \zeta_i| \end{aligned}$$

$$= \zeta_1 + n - 1 + ln|\frac{\prod_{i=1}^{n} \zeta_i}{\zeta_1}|$$
$$= \zeta_1 + n - 1 + ln\,|\prod_{i=1}^{n} \zeta_i| - ln\,\zeta_1$$
$$= \zeta_1 + n - 1 + ln\,|det(A_{ISI}(G)| - ln\,\zeta_1.$$

∎

5 Relation between E_{ISI} and some other topological indices

By using the Gershgorin disc theorem, the following result follows.

Let G be a graph of order n and size m. Then

$$E_{ISI}(G) \leq ISI(G).$$

Theorem 5.1. Let G be a graph of order n and size m. Then

$$E_{ISI}(G) \leq \frac{1}{2\delta} M_2.$$

Proof: Let G be a graph of order n and size m. Then from Observation 5,

$$E_{ISI}(G) \leq \sum_{v_iv_j \in E} \frac{d_id_j}{d_i + d_j} \leq \frac{1}{2\delta} \sum_{v_iv_j \in E} d_id_j.$$

Thus,

$$E_{ISI} \leq \frac{1}{2\delta} M_2.$$

∎

Theorem 5.2. Let G be a graph of order n and size m. Then

$$E_{ISI}(G) \leq \frac{\Delta^2}{2} H(G).$$

Proof: Let G be a graph of order n and size m. Then from Observation 5,

$$E_{ISI}(G) \leq \sum_{v_iv_j \in E} \frac{d_id_j}{d_i + d_j} \leq \Delta^2 \sum_{v_iv_j \in E} \frac{1}{d_i + d_j}.$$

Hence,

$$E_{ISI} \leq \frac{\Delta^2}{2} H(G).$$

∎

Theorem 5.3. Let G be a graph of order n and size m. Then

$$E_{ISI}(G) \geq \sqrt{\frac{1}{2\delta^2} R_2(G) + n(n-1)det(A_{ISI})},$$

where $R_2(G)$ is the general product-connectivity index with $\alpha = 2$.

Proof: Let G be a graph of order n and size m. By using the Arithmetic mean, Geometric mean inequality,

$$\begin{aligned} E_{ISI}^2 &\geq 2 \sum_{v_i v_j \in E} \frac{(d_i d_j)^2}{(d_i + d_j)^2} + n(n-1) \prod_{i=1}^{n} \zeta_i \\ &\geq 2 \frac{1}{4\delta^2} \sum_{v_i v_j \in E} (d_i d_j)^2 + n(n-1) \prod_{i=1}^{n} \zeta_i \\ &= \frac{1}{2\delta^2} R_2(G) + n(n-1) \prod_{i=1}^{n} \zeta_i. \end{aligned}$$

Thus, the result follows. ∎

References

[1] R. B. Bapat, *Graphs and Matricies*, Hindustan Book Agency, 2011.

[2] B. Bollob$\acute{a}$s and P. Erdos, Graphs of extremal weights, *Ars Comb*, 50 (1998), 225-233.

[3] D. Cherny, T. Denton, R. Thomas and A. Waldron, *Linear Algebra*, Edited by Katrina Glaeser and Travis Scrimshaw First Edition. Davis California, 2013.

[4] L. Collatz and U. Sinogwitz, Spectren endlicher grafen, *Abh. Math. Sem. Univ. Hamburg*, 21 (1957), 63-77.

[5] C. S. Edwards and C. H. Elphick, Lower bounds for the clique and the chromatic numbers of a graph, *Discr. Appl. Math.*, 5 (1983), 51-64.

[6] F. Falahati-Nezhad, M. Azari and T. Došlić, Sharp bounds on the inverse sum indeg index, *Discrete Appl. Maths.*, in press.

[7] I. Gutman, X. Li and J. Zhang, Graph energy, (Ed-s: M. Dehmer, F. Emmert), Streib. Analysis of Complex Networks, From Biology to Linguistics, *Wiley-VCH, Weinheim*, (2009), 145-174.

[8] I. Gutman, The energy of a graph, *Ber. Math. Stat. Sekt. Forsch. Graz*, 103 (1978), 1-22.

[9] I. Gutman, B. Ruscic, N. Trinajstic and C. F. Wilcox, Graph theory and mulicular orbitals, *XII. Acyclic Polyenes, J. Chem. Phys*, 62 (1975), 3399-3405.

[10] I. Gutman and N. Trinajstic, Graph theory and molecular orbitals. Total π-electron energy of alternant hydrocarbons, *Chem. Phys. Lett.*, 17 (1972), 535-538.

[11] F. Harary, *Graph Theory*, Addison-Wesley Publishing Co., Reading, Mass. Menlo Park, Calif. London, 1969.

[12] R. A. Horn and C.R. Johnson, *Matrix Analysis*, Cambridge Univ. Press New York, (1990).

[13] X. Li and I. Gutman, *Mathematical Aspects of Randić-Type Molecular Structure Descriptors*, University of Kragujvac, Kragujvac, 2006.

[14] J. Li and W. C. Shiu, The harmonic index of a graph, *Rocky Mountain J. Maths.*, 44 (2014), 1607-1620.

[15] K. Pattabiraman, *Inverse sum indeg index of graphs*, AKCE Int. J. Graphs Comb., 2 (2018), 155-167.

[16] J. Sedlar, D. Stevanović and A. Vasilyev, On the inverse sum indeg index, *Discr. Appl. Maths.*, 184 (2015), 202-212.

[17] B. Sharada, M. I. Sowaity and A. M. Naji, The block energy of a graph, *Online J. Analytic Comb.*, 14 (2019), 1-13.

[18] B. Sharada and M. I. Sowaity, On the sum-eccentricity energy of a graph, *Int. J. Appl. Graph Theory*, 2 (2018), 1-8.

[19] B. Sharada, M. I. Sowaity and I. Gutman, Laplacian sum-eccentricity energy of a graph, *Math. Interdisc. Res.*, 2 (2017), 209-220.

[20] M. I. Sowaity, B. Sharada and A. M. Naji, The eccentricity extended energy of a graph, *Proceedings Jangjeon Math. Society*, 2 (2019), 337-347.

[21] M. I. Sowaity and B. Sharada, The sum-eccentricity energy of a graph, *Int. J. Recent Innovation Trends Comp. Comm.*, 5 (2017), 293-304.

[22] D. Vakoćević and M. Gaśperove, Bound additive modeling 1. Adriatic indices, *Croat. Chem. Acta*, 83 (2010), 243-260.

[23] L. Zhong, Tje harmonic index on graphs, *Appl. Math. Lett.*, 25 (2012), 561-566.