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Forcing Edge Triangle Free Detour Number of an Edge Triangle Free Detour Graph

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Abstract

For any two vertices u and v in a connected graph $G = (V, E)$, the $u-v$ path P is called a $u-v$ triangle free path if no three vertices of P induce a triangle. The triangle free detour distance $D_{\triangle f}(u, v)$ is the length of a longest $u-v$ triangle free path in G. A $u-v$ path of length $D_{\triangle f}(u, v)$ is called a $u-v$ triangle free detour. A set $S \subseteq V$ is called an edge triangle free detour set of G if every edge of G lies on a triangle free detour joining a pair of vertices of S. The edge triangle free detour number $edn_{\triangle f}(G)$ of G is the minimum order of its edge triangle free detour sets and any edge triangle free detour set of order $edn_{\triangle f}(G)$ is called an edge triangle free detour basis of G. A graph G is called an edge triangle free detour graph if it has an edge triangle free detour set. Let G be an edge triangle free detour graph and S an edge triangle free detour basis of G. A subset $T \subseteq S$ is called a forcing subset for S if S is the unique edge triangle free detour basis containing T. A forcing subset for S of minimum order is a minimum forcing subset of S. The forcing edge triangle free detour number of G is $fedn_{\triangle f}(G) = min\{fedn_{\triangle f}(S)\}$, where the minimum is taken over all edge triangle free detour bases S in G. We determine bounds for it and find the forcing edge triangle free detour number of certain classes of graphs.

Key words: edge triangle free detour set, edge triangle free detour number, forcing edge triangle free detour number.

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1 Introduction

By a graph $G = (V, E)$, we mean a finite undirected connected simple graph. For basic definitions and terminologies, we refer to Chartrand et al. [5]. The neighbourhood of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v. A vertex v is an extreme vertex if the subgraph induced by its neighbors is complete.

The concept of triangle free detour distance was introduced by Keerthi Asir and Athisayanathan [6]. A path P is called a triangle free path if no three vertices of P induce a triangle. For vertices u and v in a connected graph G, the triangle free detour distance $D_{\triangle f}(u, v)$ is the length of a longest $u - v$ triangle free path in G. A $u - v$ path of length $D_{\triangle f}(u, v)$ is called a $u - v$ triangle free detour.

The concept of triangle free detour number was introduced by Sethu Ramalingam et al. [8]. A set $S \subseteq V$ is called triangle free detour set of G if every vertex of G lies on a triangle free detour joining a pair of vertices of S. The triangle free detour number $dn_{\triangle f}(G)$ of G is the minimum order of its triangle free detour sets and any triangle free detour set of order $dn_{\triangle f}(G)$ is called a triangle free detour basis of G.

The concept of edge triangle free detour number was introduced by Athisayanathan et al. [1]. A set $S \subseteq V$ is called edge triangle free detour set of G if every edge of G lies on a triangle free detour joining a pair of vertices of S. The edge triangle free detour number $edn_{\triangle f}(G)$ of G is the minimum order of its edge triangle free detour sets and any edge triangle free detour set of order $edn_{\triangle f}(G)$ is called an edge triangle free detour basis of G.

In this paper, we introduce a forcing edge triangle free detour number of an edge triangle free detour graph in a connected graph G. Throughout this paper, G denotes a connected graph with atleast two vertices.

The following theorems will be used in the sequel.

Theorem 1.1. Every extreme vertex of a connected graph G belongs to every edge triangle free detour set of G.

Theorem 1.2. If T is a tree with k end-vertices, then $edn_{\triangle f}(T) = k$.

Theorem 1.3. For any connected graph G of order n, $2 \leq edn_{\triangle f}(G) \leq n$.

Theorem 1.4. For the complete graph K_n, $edn_{\triangle f}(G) = n$.

Theorem 1.5. No cut vertex of a connected graph G belongs to any edge triangle free detour basis of G.

Theorem 1.6. Let G be a complete bipartite graph $K_{n,m}(2 \leq n \leq m)$. Then a set $S \subseteq V$ is an edge triangle free detour basis of G if and only if S consists of any two vertices of G.

2 Forcing Edge Triangle Free Detour Number of an Edge Triangle Free Detour Graph

Definition 2.1. Let G be an edge triangle free detour graph and S an edge triangle free detour basis of G. A subset $T \subseteq S$ is called a forcing subset for S if S is the unique edge triangle free detour basis containing T. A forcing subset for S of minimum order is a minimum forcing subset of S. The forcing edge triangle free detour number of G is $fedn_{\triangle f}(G) = min\{fedn_{\triangle f}(S)\}$, where the minimum is taken over all edge triangle free detour bases S in G.

Example 2.2. For the graph G given in Figure 2.1, $S_1 = \{z, w, v, x\}$, $S_2 = \{z, w, v, u\}$, $S_3 = \{z, w, u, x\}$ and $S_4 = \{z, u, v, x\}$ are the edge triangle free detour bases of G, so that $edn_{\triangle f}(G) = 4$ and $fedn_{\triangle f}(G) = 3$.

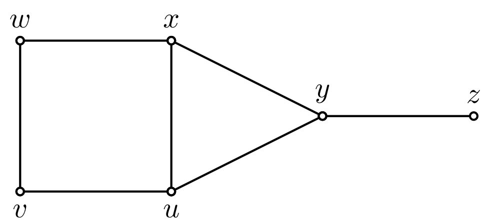

Figure 2.1 : G

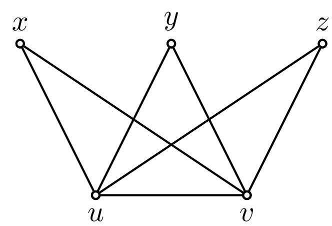

Figure 2.2 : G

For the graph G given in Figure 2.2, $S_5 = \{u, v, x, y, z\}$ is the unique edge triangle free detour basis of G and so $fedn_{\triangle f}(G) = 0$.

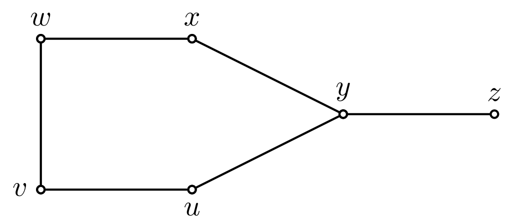

Figure 2.3 : G

For the graph G given in Figure 2.3, $S_6 = \{z, x, u\}$, $S_7 = \{z, w, u\}$, $S_8 = \{z, v, x\}$ and $S_9 = \{z, v, w\}$ are an edge triangle free detour bases of G so that $fedn_{\triangle f}(G) = 2$.

The following theorem immediately from the definitions of an edge triangle free detour number and forcing edge triangle free detour number of a connected graph G.

Theorem 2.3. For any edge triangle free detour graph G, $0 \leq fedn_{\triangle f}(G) \leq edn_{\triangle f}(G)$.

Proof: It is clear from the definition of $fedn_{\triangle f}(G)$ that $fedn_{\triangle f}(G) \geq 0$. Let S be any edge triangle free detour basis of G. Since $fedn_{\triangle f}(S) \leq edn_{\triangle f}(G)$ and since $fedn_{\triangle f}(G) = min\{fedn_{\triangle f}(S) : S$ is an edge triangle free detour basis of $G\}$, it follows that $fedn_{\triangle f}(G) \leq edn_{\triangle f}(G)$. Thus $0 \leq fedn_{\triangle f}(G) \leq edn_{\triangle f}(G)$. ■

Remark 2.4. The bounds in Theorem 2.3 are sharp. For the graph G given in Figure 2.2, $fedn_{\triangle f}(G) = 0$. For the odd cycle $C_n (n \geq 5)$, $fedn_{\triangle f}(G) = edn_{\triangle f}(G) = 3$. Also, the inequalities in Theorem 2.3 can be strict. For the graph G given in Figure 2.1, $edn_{\triangle f}(G) = 4$ and $fedn_{\triangle f}(G) = 2$. Thus $0 < fedn_{\triangle f}(G) < edn_{\triangle f}(G)$.

Theorem 2.5. Let G be an edge triangle free detour graph. Then
(a) $fedn_{\triangle f}(G) = 0$ if and only if G has a unique edge triangle free detour basis.
(b) $fedn_{\triangle f}(G) = 1$ if and only if G has at least two edge triangle free detour bases, one of which is a unique edge triangle free detour basis containing one of its elements.
(c) $fedn_{\triangle f}(G) = edn_{\triangle f}(G)$ if and only if no edge triangle free detour basis of G is the unique edge triangle free detour basis containing any of its proper subsets.

Proof: (a) Let $fedn_{\triangle f}(G) = 0$. Then, by definition, $fedn_{\triangle f}(S) = 0$ for some edge triangle free detour basis S of G so that empty set Φ is the minimum forcing subset of S.

Since the empty set Φ is a subset of every set, it follows that S is the unique edge triangle free detour basis of G. The converse is clear.

(b) Let $fedn_{\triangle f}(G) = 1$. Then by (a), G has at least two edge triangle free detour bases. Also, since $fedn_{\triangle f}(G) = 1$, there is a singleton subset T of an edge triangle free detour basis S of G such that T is not a subset of any other edge triangle free detour basis of G. Thus S is the unique edge triangle free detour basis containing one of its elements. The converse is clear.

(c) Let $fedn_{\triangle f}(G) = edn_{\triangle f}(G)$. Then $fedn_{\triangle f}(S) = edn_{\triangle f}(G)$ for every edge triangle free detour basis S in G. Also by Theorem 1.3, $edn_{\triangle f}(G) \geq 2$ and hence $fedn_{\triangle f}(G) \geq 2$. Then by (b), G has at least two edge triangle free detour bases and so the empty set Φ is not a forcing subset of any edge triangle free detour basis of G. Since $fedn_{\triangle f}(S) = edn_{\triangle f}(G)$, no proper subset of S is a forcing subset of S. Thus no edge triangle free detour basis of G is the unique edge triangle free detour basis containing any of its proper subsets.

Conversely, the data implies that G contains more than one edge triangle free detour basis and no subset of any edge triangle free detour basis S other than S is a forcing subset for S. Hence it follows that $fedn_{\triangle f}(G) = edn_{\triangle f}(G)$. ■

We observe that if G has a unique edge triangle free detour basis S, then every vertex in S is an edge triangle free detour vertex of G. Also, if x is an extreme vertex of G, then x is an edge triangle free detour vertex of G. For the graph G given in Figure 2.2, $S = \{u, v, x, y, z\}$ is the only edge triangle free detour basis of G so that all the vertices of S are the edge triangle free detour vertices of G.

Theorem 2.6. Let G be an edge triangle free detour graph and let Im be the set of relative complements of the minimum forcing subsets in their respective edge triangle free detour basis in G. Then $\cap_{F\in \mathrm{Im}} F$ is the set of edge triangle free detour vertices of G.

Proof: Let W be the set of all edge triangle free detour vertices of G. We claim that $W = \cap_{F\in \mathrm{Im}} F$. Let $v \in W$. Then v is an edge triangle free detour vertex of G so that v belongs to every edge triangle free detour basis S of G. Let $T \subseteq S$ be any minimum forcing subset for any edge triangle free detour basis S of G. We claim that $v \notin T$. If $v \in T$, then $T' = T - \{v\}$ is a proper subset of T such that S is the unique edge triangle free detour basis containing T' so that T' is a forcing subset for S with $|T'| < |T|$, which is a contradiction to T a minimum forcing subset for S. Thus $v \notin T$ and so $v \in F$, where F is the relative complement of T in S. Hence $v \in \cap_{F\in \mathrm{Im}}F$ so that $W \subseteq \cap_{F\in \mathrm{Im}}F$.

Conversely, let $v \in \cap_{F\in \mathrm{Im}}F$. Then v belongs to the relative complement of T in S for every T and every S such that $T \subseteq S$, where T is a minimum forcing subset for S. Since F is the relative complement of T in S,$F \subseteq S$ and so $v \in S$ for every S so that v is an edge triangle free detour vertex of G. Thus $v \in W$ and so $\cap_{F\in \mathrm{Im}}F \subseteq W$. ■

Theorem 2.7. Let G be an edge triangle free detour graph and S be any edge triangle free detour basis of G. Then

(a) No edge triangle free detour vertex of G belongs to any minimum forcing set of S.

(b) No cut-vertex of G belongs to any minimum forcing subset of G.

Proof: (a) The proof is contained in the proof of the first part of Theorem 2.6.

(b) This follows from Theorem 1.5. ■

Theorem 2.8. Let G be an edge triangle free detour graph and let M be the set of all edge triangle free detour vertices of G. Then $fedn_{\triangle f}(G) \leq edn_{\triangle f}(G) - |M|$.

Proof: Let S be any edge triangle free detour basis of G. Then $edn_{\triangle f}(G) = |S|$,$M \subseteq S$ and S is the unique edge triangle free detour basis containing $S - M$. Thus $fedn_{\triangle f}(G) \leq |S - M| = |S| - |M| = edn_{\triangle f}(G) - |M|$. ■

Corollary 2.9. If G is a connected graph with l extreme vertices, then $fedn_{\triangle f}(G) \leq edn_{\triangle f}(G) - l$.

Remark 2.10. The bound in Theorem 2.8 is sharp. For the graph G given in Figure 2.1, $fedn_{\triangle f}(G) = 3$, $|M| = 1$ and $edn_{\triangle f}(G) = 4$. Also, the inequality in Theorem 2.8 can be strict. For the cycle C_4, $edn_{\triangle f}(C_4) = 2$, $|M| = 0$ and $fedn_{\triangle f}(C_4) = 1$. Thus $fedn_{\triangle f}(G) < edn_{\triangle f}(G) - |M|$.

In the following theorem we determine $fedn_{\triangle f}(G)$ for certain graphs G.

Theorem 2.11. Let G be a connected graph of order n. Then

(a) If G is the complete bipartite graph $K_{n,m}(2 \leq m \leq n)$, then $edn_{\triangle f}(G) = fedn_{\triangle f}(G) = 2$.

(b) If G is the cycle $C_n(n \geq 4)$, then

(i) $edn_{\triangle f}(G) = 2$ and $fedn_{\triangle f}(G) = 1$ for n is even.

(ii) $edn_{\triangle f}(G) = fedn_{\triangle f}(G) = 3$ for n is odd.
(c) If G is the tree with k end-vertices, then $edn_{\triangle f}(G) = k$ and
$fedn_{\triangle f}(G) = 0$.
(d) If G is the complete graph K_n, then $edn_{\triangle f}(G) = n$ and $fedn_{\triangle f}(G) = 0$.

Proof: (a) By Theorem 1.6, a set S of vertices is an edge triangle free detour basis if and only if S consists of any two vertices of G. For each vertex v in G there are two or more vertices adjacent with v. Thus the vertex v belongs to more than one edge triangle free detour basis of G. Hence it follows that no set consisting of a single vertex is a forcing subset for any edge triangle free detour basis of G. Thus the result follows.

(b)(i) Let n be even. Then a set $S = \{u, v\}$ is an edge triangle free detour basis of G if and only if u and v are antipodal vertices in G. Clearly, $edn_{\triangle f}(G) = 2$ and each vertex u in G belongs to exactly only one edge triangle free detour basis of C_n. So it follows that every set consisting of a single vertex of G is a forcing subset for an edge triangle free detour basis of G and hence $fedn_{\triangle f}(G) = 1$.
(ii) Let n be odd. Then it can be easily verify that $edn_{\triangle f}(G) = 3$. A set S is an edge triangle free detour basis of G if and only if S consisting of any three vertices of G. Since $n \geq 4$, no subset of V of cardinality 0, 1 and 2 is a forcing subset for any edge triangle free detour basis of C_n. Therefore by theorem 2.5 (c), $fedn_{\triangle f}(G) = 3$.
(c) By Theorem 1.2, $edn_{\triangle f}(G) = k$. Since the set of all end-vertices of a tree is the unique edge triangle free detour basis, the result follows from Theorem 2.5(a).

(d) For K_n, it follows from Theorem 1.4 that the set of all vertices of G is the unique edge triangle free detour basis of G. It follows from Theorem 2.5(a) that $fedn_{\triangle f}(G) = 0$. ■

The following theorem gives a realization result.

Theorem 2.12. For any two positive integers a,b with $0 \leq a \leq b$ and $b \geq 2$, there is an edge triangle free detour graph G such that $fedn_{\triangle f}(G) = a$, $edn_{\triangle f}(G) = b$.

Proof: Case 1. $a = 0$. For each $b \geq 2$, let G be a tree with b end-vertices. Then $fedn_{\triangle f}(G) = 0$ and $edn_{\triangle f}(G) = b$ by Theorem 2.11(c).

Case 2. $a \geq 1$. For each $i(1 \leq i \leq a)$, let $F_i : u_i, v_i, w_i, x_i, u_i$ be the cycle of order 4 and let $H = K_{1,b-a}$ be a star at v whose set of end-vertices is $\{z_1, z_2, ..., z_{b-a}\}$. Let G be the graph obtained by joining the central vertex v of H to both vertices u_i,w_i of each $F_i(1 \leq i \leq a)$. Clearly the graph G is connected and is shown in Figure 2.4. Let $W = \{z_1, z_2, ..., z_{b-a}\}$ be the set of all $(b - a)$ end-vertices of G.

First, we show that $edn_{\triangle f}(G) = b$. Then by Theorems 1.1 and 1.5 every edge triangle free detour basis contains W and at least one vertex from each $F_i(1 \leq i \leq a)$. Thus $edn_{\triangle f}(G) \geq (b - a) + a = b$. On the other hand, since the set $S_1 = W \cup \{v_1, v_2, ..., v_a\}$ is an edge triangle free detour set of G, it follows that $dn_{\triangle f}(G) \leq |S_1| = b$. Therefore $edn_{\triangle f}(G) = b$.

Next we show that $fedn_{\triangle f}(G) = a$. It is clear that W is the set of all edge triangle free detour vertices of G. Hence it follows from Theorem 2.8 that $fedn_{\triangle f}(G) \leq edn_{\triangle f}(G) - |W| = b - (b - a) = a$. Now, since $edn_{\triangle f}(G) = b$, it is easily seen that a set S is an edge triangle free detour basis of G if and only if S is of the form $S = W \cup \{y_1, y_2, ..., y_a\}$, where $y_i \in \{v_i, x_i\} \subseteq V(F_i)(1 \leq i \leq a)$. Let T be a subset of S with $|T| < a$. Then there is a vertex $y_j(1 \leq j \leq a)$ such that $y_j \notin T$. Let $s_j \in \{v_j, x_j\} \subseteq V(F_j)$ disjoint from y_j. Then $S' = (S - \{y_j\}) \cup \{s_j\}$ is an edge triangle free detour basis that contains T. Thus S is not the unique edge triangle free detour basis containing T. Thus $fedn_{\triangle f}(S) \geq a$. Since this is true for all edge triangle free detour basis of G, it follows that $fedn_{\triangle f}(G) \geq a$ and so that $fedn_{\triangle f}(G) = a$. ∎

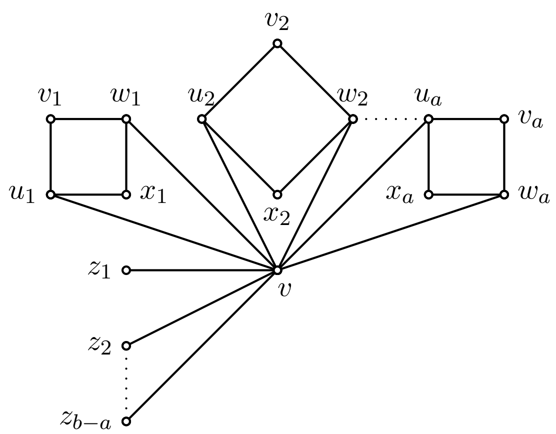

Figure 2.4 : G

3 Acknowledgment

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