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Cycle Related Subset Cordial Graphs

D. K. Nathan and *K. Nagarajan*
PG and Research Department of Mathematics
Sri S.R.N.M.College, Sattur - 626 203
Tamil Nadu, India.
dknathansrnmcstr@gmail.com
k_nagarajan_srnmc@yahoo.co.in

Abstract

A subset cordial labeling of a graph G with vertex set V is an injection f from V to the power set of $\{1, 2, \ldots, n\}$ such that an edge uv is assigned the label 1 if either$f(u)$ is a subset of $f(v)$ or $f(v)$ is a subset of $f(u)$ and 0 otherwise. Then the number of edges labeled 0 and 1 differ by atmost 1. If a graph has a subset cordial labeling, then it is called a subset cordial graph. In this paper, we prove that some cycle related graphs such as splitting graph of cycle, identification of cycle with a graph, wheel with two centers, flower graph and dragon are subset cordial.

Keywords: Cordial labeling, subset cordial labeling, subset cordial graphs, splitting graph.

2010 Mathematics Subject Classification : 05C78.

1 Introduction

By a graph, we mean a finite, undirected graph without loops and multiple edges and for terms not defined here, we refer to Harary [6].

We recall some well known definitions and results which are useful for the present study [7].

Let $X = \{1, 2, \ldots, n\}$ be a set and $\wp(X)$ be a collection of all subsets of X, called the power set of X. If A is a subset of B, we denote it by $A \subset B$, otherwise by $A \not\subset B$. Note that $\wp(X)$ contains 2^n subsets.

* Corresponding Author: *D. K. Nathan*
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Graph labeling [5] is a strong communication between Algebra [7] and structure of graphs [6]. By combining the set theory concept in Algebra and cordial labeling concept in graph labeling, we introduced a new concept called **subset cordial labeling** [8]. In this paper, we prove some cycle related graphs such as splitting graph of cycle, identification of cycle with a graph and the like are subset cordial.

A vertex labeling [5] of a graph G is an assignment f of labels to the vertices of G that induces each edge uv a label depending on the vertex label $f(u)$ and $f(v)$. Graceful and harmonious labeling are two well known labelings. Cordial labeling is a variation of both graceful and harmonious labeling [3].

Definition 1.1. Let $G = (V, E)$ be a graph. A mapping $f : V(G) \to \{0, 1\}$ is called binary vertex labeling of G and $f(v)$ is called the label of the vertex v of G under f.

For an edge $e = uv$, the induced edge labeling $f^* : E(G) \to \{0, 1\}$ is given by $f^*(e) = |f(u) - f(v)|$. Let $v_f(0)$ and $v_f(1)$ be the number of vertices of G having labels 0 and 1 respectively under f. Let $e_f(0)$ and $e_f(1)$ be the number of edges having labels 0 and 1 respectively under f^*.

Definition 1.2. [3]

A binary vertex labeling of a graph G is called a cordial labeling, if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph G is cordial, if it admits cordial labeling.

Sundaram, Ponraj and Somasundaram [10] have introduced the notion of prime cordial labeling and have proved that some graphs are prime cordial. R. Varatharajan, S. Navaneethakrishnan and K. Nagarajan [13] introduced
divisor cordial labeling and they have proved that some graphs are divisor cordial[14].

Definition 1.3. [10] A prime cordial labeling of a graph G with vertex set V is a bijection $f : V \longrightarrow \{1, 2, \ldots, |V|\}$ such that if each edge uv is assigned the label 1 if $\gcd(f(u), f(v)) = 1$ and 0 if $\gcd(f(u), f(v)) > 1$; Then the number of edges labeled with 1 and the number of edges labeled with 0 differ by at most 1.

Definition 1.4. [13] Let G=(V, E) be a simple graph and $f : V \to \{1, 2, \ldots |V|\}$ be a bijection. For each edge uv, assign the label 1 if either $f(u) \mid f(v)$ or $f(v) \mid f(u)$ and the label 0 if $f(u)(v)$. f is called a divisor cordial labeling if $|e_f(0) - e_f(1)| \leq 1$.

A graph with a divisor cordial labeling is called a divisor cordial graph.

Using set theory concepts, some authors have established set sequential graphs[2] and set graceful labeling[2]. Motivated by the concepts of prime cordial labeling and divisor cordial labeling, we introduced a new cordial labeling called **subset cordial labeling** [8] .

Definition 1.5. [8] Let $X = \{1, 2, \ldots, n\}$ be a set. Let $G = (V, E)$ be a simple (p, q)-graph and $f : V \to \wp(X)$ be an injection. Let $2^{n-1} < p \leq 2^n$. For each edge uv, assign label 1 if either $f(u) \subset f(v)$ or $f(v) \subset f(u)$ and assign 0 if either $f(u) \not\subset f(v)$ or $f(v) \not\subset f(u)$. f is called a subset cordial labeling if $|e_f(0) - e_f(1)| \leq 1$.

A graph is called a **subset cordial graph** if it has a subset cordial labeling. Here $\wp(X)$ denotes the power set of X.

Example 1.6. Consider the following graph G. Take $X = \{1, 2, 3\}$.

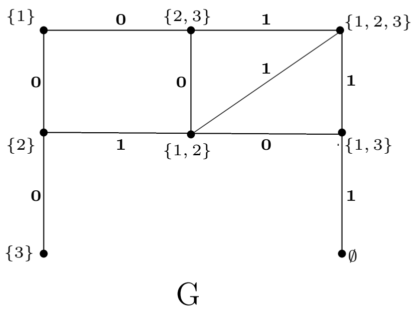


Figure 1: Subset cordial labeling of a given graph G.

Here $e_f(0) = 5 = e_f(1)$ *and* $|e_f(1) - e_f(0)| = 0$. *Thus* G *is subset cordial.*

Remark 1.7. If $2^{n-1} < p < 2^n$ and $X = \{1, 2, \ldots, n\}$, we have more number of subsets than vertices. We can easily label the vertices with $|e_f(0) - e_f(1)| \leq 1$. If the number of vertices is 2^n, then the result will be very strong.

Example 1.8. *In Example 1.6, by removing the vertices labeled* $\{3\}$ *and* $\emptyset$, *we get the following subset cordial graph.*

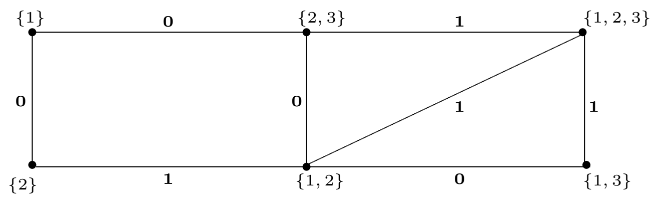


Figure 2: Subset cordial labeling of the graph given in Figure1 by removing the vertices labeled $\{3\}$ and $\emptyset$.

In [8] and [9], we proved the following results, which will be used in the subsequent Theorems.

Theorem 1.9. [8] The path P_{2^n} is subset cordial.

Theorem 1.10. [8] The cycle C_{2^n} is subset cordial.

Theorem 1.11. [8] The wheel W_{2^n} is subset cordial for $n \geq 3$.

Theorem 1.12. [9]The star graph $K_{1,q}$ is subset cordial.

2 Main Results

In this paper, we prove some cycle related graphs such as dragon, wheel with two centers, one point union of cycles , identification of cycle with a graph and flower graph which are subset cordial.

First, we prove that dragons are subset cordial.

Definition 2.1. A dragon is obtained by joining a vertex of a cycle with a pendant vertex of a path.

Theorem 2.2. Dragon D is subset cordial.

Proof: Let $X = \{1, 2, \ldots, n\}$ and let $p = 2^n$. Then $q = 2^n$ for a dragon D.

Let P: $v_1 v_2 \ldots v_{m-1}$ be a path and C: $v_m v_{m+1} \ldots v_{2^n}$ be a cycle. Let the dragon be obtained by making v_{m-1} and v_m as adjacent and v_m as the common vertex to the path P and the cycle C. Then the vertex set of the dragon D is $\{v_1, v_2, \ldots, v_{m-1}, v_m, v_{m+1}, \ldots, v_{2^n}\}$.

Now, we arrange all the subsets of X in the following pattern (illustrated for $X = \{1, 2, 3, 4\}$) and label the vertices of D by the given direction. We note that the vertices v_1 and v_{2^n} are labeled with the improper subsets $\emptyset$ and $\{1, 2, \ldots, n\}$. The edges labeled with 1 are obtained by labeling the end vertices by the subsets $\{n_1, n_2, \ldots, n_k\}$ and $\{1, n_1, n_2, \ldots, n_k\}$ such that

(i) $1 < n_1 < n_2 < \ldots < n_k$

(ii) $1 \leq k \leq n - 1$

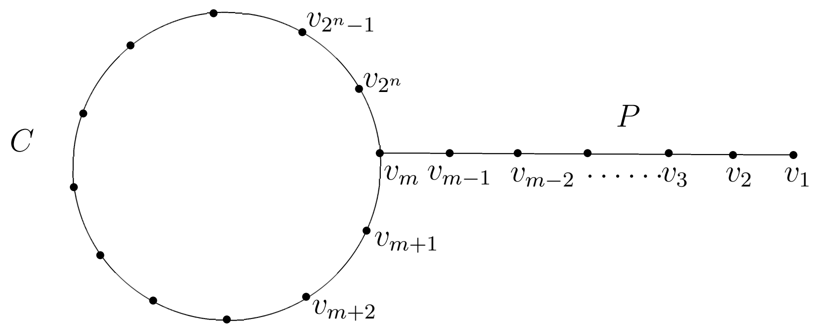


Figure 3: Dragon D obtained from the path P and the cycle C.

Thus, we see that the edges are labeled with 1 and 0 alternately starting and ending with the label 1. We also see that the vertex v_{2^n} labeled with improper subset $\{1, 2, \ldots n\}$ is adjacent to v_m which can be any subset. Then the edge $v_m v_{2^m}$ is clearly getting label 1.

So, by construction, we have $e_f(1) = 2^{n-1}+1$ and $e_f(0) = 2^{n-1}-1$. Then $|e_f(1) - e_f(0)| = 2$.

Now, the subset cordiality of D is obtained by the following changes of labeled vertices except the improper subset labels.

Consider any three consecutive vertices labeled with the subsets B, C and D such that$B \not\subseteq C \subset D$. Then the edge labeling pattern is given below.

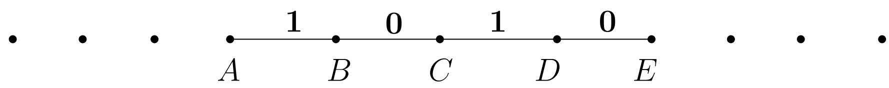


Figure 4: Edge labeling pattern.

Note that A and E are subsets such that $A \subset B$ but D is not a subset of E. Now, we change the labeling pattern of vertices as follows.

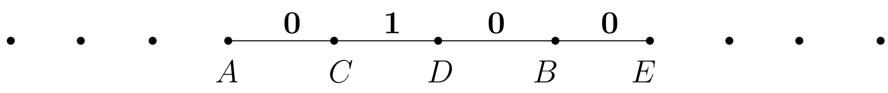


Figure 5: Change of edge labeling pattern.

Now, we have $e_f(1) = 2^{n-1} + 1 - 1 = 2^{n-1}$ and $e_f(0) = 2^{n-1} - 1 + 1 = 2^{n-1}$ and so $|e_f(1) - e_f(0)| = 0$. Thus D is subset cordial for $p = 2^n$. Next, we prove that D is subset cordial for $2^{n-1} < p < 2^n$. In the above labeling for $p = 2^n$, if we remove the 2^{n-1} vertices one by one starting from v_p, then we have $|e_f(0) - e_f(1)| \leq 1$. Then the result is obtained. Hence the dragon D is subset cordial for any p.

Example 2.3. *Take* $X = \{1, 2, 3, 4\}$. *Then* $p = 2^4 = 16$. *Consider the following dragon graph with* 16 *vertices.*

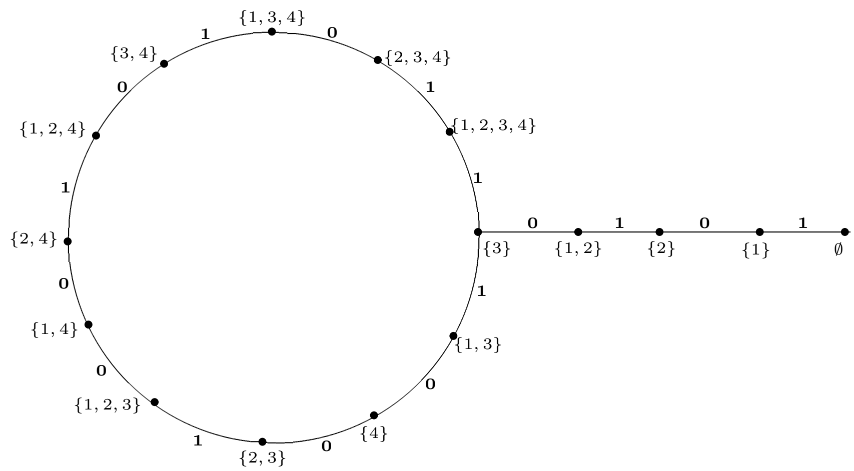


Figure 6: *Subset cordial labeling of dragon D.*

Here $e_f(0) = e_f(1) = 8$ *and* $|e_f(0) - e_f(1))| = 0$. *Thus the dragon D is subset cordial.*

Note 2.4. *We have already proved that wheel graph is subset cordial in Theorem 1.11. If we label the subset* $\{1\}$ *to the center of the wheel, then the subset cordiality remains unchanged. Using this observation , we can prove that the wheel graph is subset cordial.*

Definition 2.5. The wheel graph W_n is defined as $K_1 + C_n$. The vertex corresponding to K_1 is called apex vertex, vertices corresponding to cycle C_n are called rim vertices and the edges corresponding to cycle C_n are called rim edges.

Lemma 2.6. Let $X = \{1, 2, \ldots, n, n+1\}$. Then the wheel graph $W_{1,2^n}$ is subset cordial.

Proof: We see that $|\wp(X)|=2^{n+1} > 2^n + 1 = p(W_{1,2^n})$. Since $W_{1,2^n} = K_1 + C_{2^n}$, the vertices of C_{2^n} are labeled by all subsets of $\{1, 2, \ldots, n\}$ as in the Theorem1.10. Then by the Theorem 1.10, C_{2^n} is subset cordial. Then for C_{2^n}, $e_f(0) = e_f(1)) = 2^{n-1}$. Now, adding $n + 1$ to each subset labeled to the vertices of C_{2^n}, then the subset cordiality of C_{2^n} remains unchanged. Now we assign $\{1\}$ to the centre of vertex of $W_{1,2^n}$. We observe that $\{1\}$ is not a subset of the subsets of $\{2, 3, \ldots, n\}$. Since the number of subsets of $\{2, 3, \ldots, n\}$ is 2^{n-1}, for spokes of $W_{1,2^n}$, we have $e_f(0) = 2^{n-1}$. Then $e_f(1) = 2^n - 2^{n-1} = 2^{n-1}$.

Hence for $W_{1,2^n}$, we have $e_f(0) = 2^{n-1} + 2^{n-1} = 2^n = e_f(1)$ and $|e_f(0) - e_f(1))| = 0$. Thus $W_{1,2^n}$ is subset cordial.

∎

Example 2.7. *Take $X = \{1, 2, 3, 4, 5\}$. The subset cordiality of $W_{1,2^4}$ is given below.*

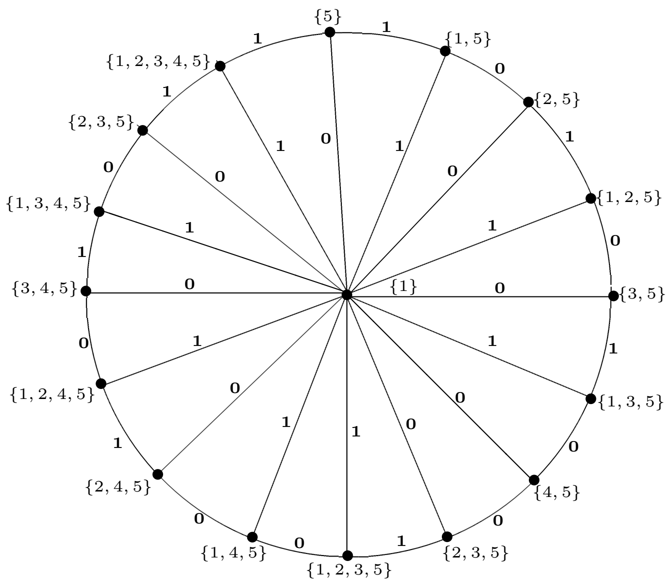


Figure 8: *Subset cordial labeling of wheel graph $W_{1,2^4}$.*

Here $e_f(0) = e_f(1) = 16$ and $|e_f(0) - e_f(1))| = 0$. Thus $W_{1,2^4}$ is subset cordial.

The Lemma 2.6 is used to prove the following Theorem.

Theorem 2.8. Let $X = \{1, 2, \ldots, n\}$ and G be any subset cordial $(2^n, q)$- graph. Then the graph $G * W_{1,2^n}$, obtained by identifying the central vertex of $W_{1,2^n}$ with that labeled $\{1\}$ in G, is also subset cordial.

Proof: Since G is a subset cordial, then $|e_f(0) - e_f(1))| \leq 1$. Already 2^n subsets of $\{1, 2, \ldots, n\}$ have been labeled to the vertices of G. Now, we construct 2^n subsets by adding a new element $n + 1$ to each subset of X and by replacing $\emptyset$ with $\{n + 1\}$. Assign these 2^n subsets to the vertices of $W_{1,2^n}$ as in the Lemma 2.6, and identify the central vertex of $W_{1,2^n}$ labeled with $\{1\}inG.Thusfor$G*W$_{1,2^n}$, it follows from the Lemma 2.6 that $|e_f(0) - e_f(1))| \leq 1$. Hence $G * W_{1,2^n}$ is subset cordial. ∎

Definition 2.9. [12] The helm H_n is the graph obtained from a wheel W_n by attaching a pendant edge to each rim vertex.

Definition 2.10. [12]The closed helm CH_n is the graph obtained by taking a helm H_n and adding edges between the pendant vertices.

Definition 2.11. [12] The flower graph Fl_p is the graph obtained from a helm $H_p$by joining each pendant vertex to the apex of the helm.

Theorem 2.12. Flower graph Fl_p is subset cordial.

Proof: Let $X = \{1, 2, \ldots, n\}$. Then the flower graph has $2^{n+1} - 1$ vertices and $2^{n+2} - 2(n - 1)$ edges. First, we label the vertices of wheel graph. We label the subset $\{1\}$ to the apex of the wheel. Label the subsets of $\{1, 2, \ldots, n\}$ except $\{1\}$ to the cycle as in the Theorem 1.10. Then from the theorem 1.11, the wheel graph is subset cordial. Add pendant edges to each of the rim vertices. Then we label the pendant vertices as follows.

If the rim vertex is labeled with the subset containing the element 1, then we label the corresponding pendant vertex by removing 1 and adding $n + 1$ to the subset labeled to the rim vertex. If the rim vertex is labeled with the subset not containing the element 1, then we label the corresponding pendant vertex by adding the elements 1 and $n + 1$ to the subset labeled to the rim vertex.

Rim vertex labeled with	corresponding pendant vertex labeled as
$\emptyset$	$\{n+1\}$
$\{1, 2, \ldots, n\}$	$\{2, 3, \ldots, n, n+1\}$
$\{1, 2, \ldots, n-1\}$	$\{2, 3, \ldots, n-1, n+1\}$
$\{2, 3, \ldots, n-2\}$	$\{1, 2, \ldots, n-2, n+1\}$
$\{3, 4, \ldots, n-1\}$	$\{1, 3, , \ldots, n-1, n+1\}$
$\{1, 3, \ldots, n-2\}$	$\{3, 4, \ldots, n-2, n+1\}$
......	......

Joining the pendant vertices to the apex of the wheel, we get the flower graph. We see that the edge joining pendant vertices and the apex get the label 0 if the corresponding spoke had the label 1 and vice versa. The edges are labeled as shown below.

Components	$e_f(0)$	$e_f(1)$
Spokes	$2^{n-1} - 1$	2^{n-1}
Rim edges	2^{n-1}	$2^{n-1} - 1$
Pendant edges of helm	$2^{n-1} - 1$	2^{n-1}
Edges joining pendant vertex and apex	2^{n-1}	$2^{n-1} - 1$
Total	$2^{n+1} - 2$	$2^{n+1} - 2$

Then the number of 0's assigned to $e_f(0)$is $2^{n+1} - 2$ and he number 1's assigned to $e_f(1)$ is $2^{n+1} - 2$. So, we have $|e_f(0) - e_f(1))| = 0$. Thus the flower graph Fl_p is subset cordial. ∎

Example 2.13. Take $X = \{1, 2, 3\}$. Consider the flower graph Fl_p with 15 vertices.

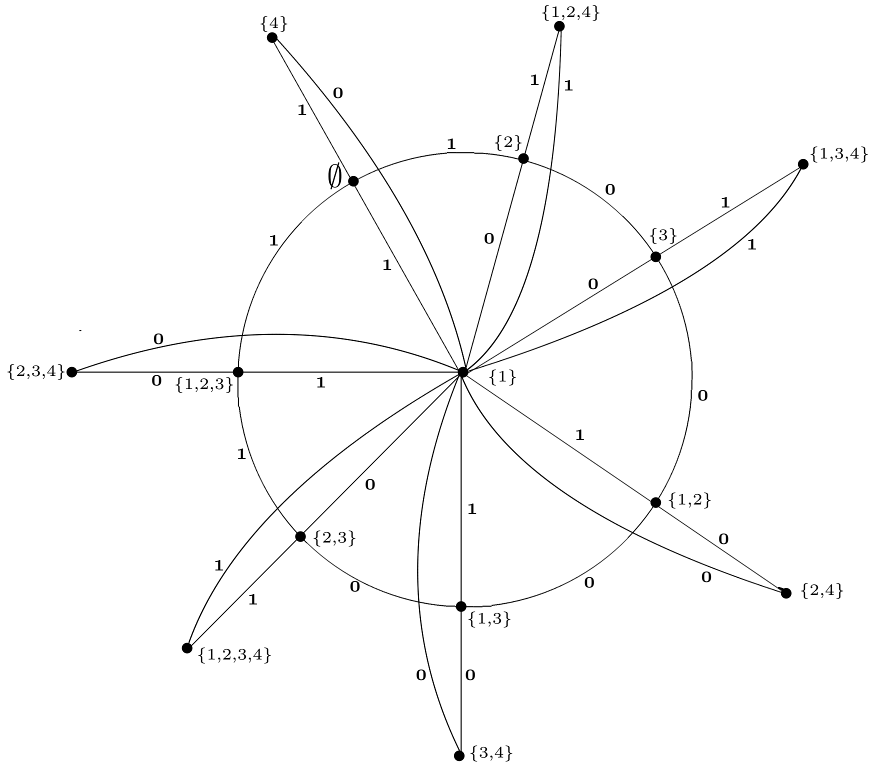


Figure 9: Subset cordial labeling of flower graph Fl_p with 15 vertices.

Here, we have $e_f(0)= e_f(1) = 14$ *and* $|e_f(0) - e_f(1))| = 0$. *Thus the flower graph* Fl_p *with* 15 *vertices is subset cordial.*

Definition 2.14. The one point union of $(t \geq 1)$ cycles, each of length n is denoted by $C_n^{(t)}$.

Now, we prove that $C_{2^n}^{(2^m)}$ *is subset cordial.*

Theorem 2.15. $C_{2^n}^{(2^m)}$ is subset cordial for $n \geq 3$ and $m \geq 1$.

Proof: Let $v_i^{(j)}(i = 1, 2, \dots, 2^n; j = 1, 2, \dots, 2^m)$ be the vertices of $C_{2^n}^{(2^m)}$ with $v_1^{(1)} = v_1^{(2)}, \dots, = v_1^{(2^m)} = v$ (say). It has $(2^n - 1)2^m$ vertices and 2^{n+m} edges.

Let $X = \{1, 2, \dots, n, n+1, \dots, n+m\}$. Then we assign the subsets of X to the vertices of $C_{2^n}^{(2^m)}$. First assign the 2^n subsets of $\{1, 2, \dots, n\}$ to the vertices of one cycle in the one point union of cycles $C_{2^n}^{(2^m)}$ as in the Theorem 1.10 with v which has the label $\emptyset$. Then the number of 1's and 0's contributed to $e_f(1)$ and $e_f(0)$ respectively is 2^{n-1} each.

Next, we construct $2^n - 1$ subsets by adding the element $n + 1$ to each subset of $\{1, 2, \dots, n\}$ except for $\emptyset$. Then we assign these $2^n - 1$ subsets to the next cycle (say, second cycle), as in the first cycle. Note that the common vertex v has already been labeled by $\emptyset$.

Then, we construct $2^n - 1$ subsets by adding the element $n+2$ to each subset of $\{1, 2, \ldots, n\}$ except for $\emptyset$. We assign these $2^n - 1$ subsets to the third cycle, as in the first cycle. Next, we construct $2^n - 1$ subsets by adding the same element $n+2$ to each subset labeled in the second cycle, except for $\emptyset$ and assigning these subsets to the vertices of the fourth cycle. Continuing this process, we observe the following.

i) Using the subsets of $\{1, 2, \ldots, n\}$, we have labeled the first cycle.

ii)Adding the element $n+1$ to the subsets of $\{1, 2, \ldots, n\}$, we have labeled the second cycle(that is 2 cycles)

iii)Adding the element $n+2$ to the subsets of $\{1, 2, \ldots, n+1\}$, we have labeled the third and fourth cycles(that is 2 cycles)

Continuing this process, atlast adding the element $n+m$ to the subsets of $\{1, 2, \ldots, n+m-1\}$, we label last 2^{m-1} cycles in the one point union of cycles. Note that the number of cycles in the one point union of cycles is

$1 + 2^0 + 2^1 + 2^2 + \ldots + 2^{m-1} = 1 + \frac{2^m - 1}{2-1} = 2^m$.

We also note that, in each cycle the number of 1's and the number of 0's contributed to $e_f(1)$ and $e_f(0)$ respectively is 2^{n-1}.

Thus $e_f(0) = 2^{m+n-1} = e_f(1)$. Hence $C_{2^n}^{(2^m)}$ is subset cordial. ∎

The Theorem 2.15 is illustrated in the following example.

Example 2.16. *Let $X = \{1, 2, 3, 4, 5\}$. The subset cordiality of $C_{2^3}^{(2^2)} = C_8^{(4)}$ is given as follows. Here we note that the subsets of $\{1, 2, 3\}$ are labeled in the first cycle. Using the element 4, we have labeled the second cycle as in the Theorem 2.15 and using the element 5, we have labeled the third and fourth cycles.*

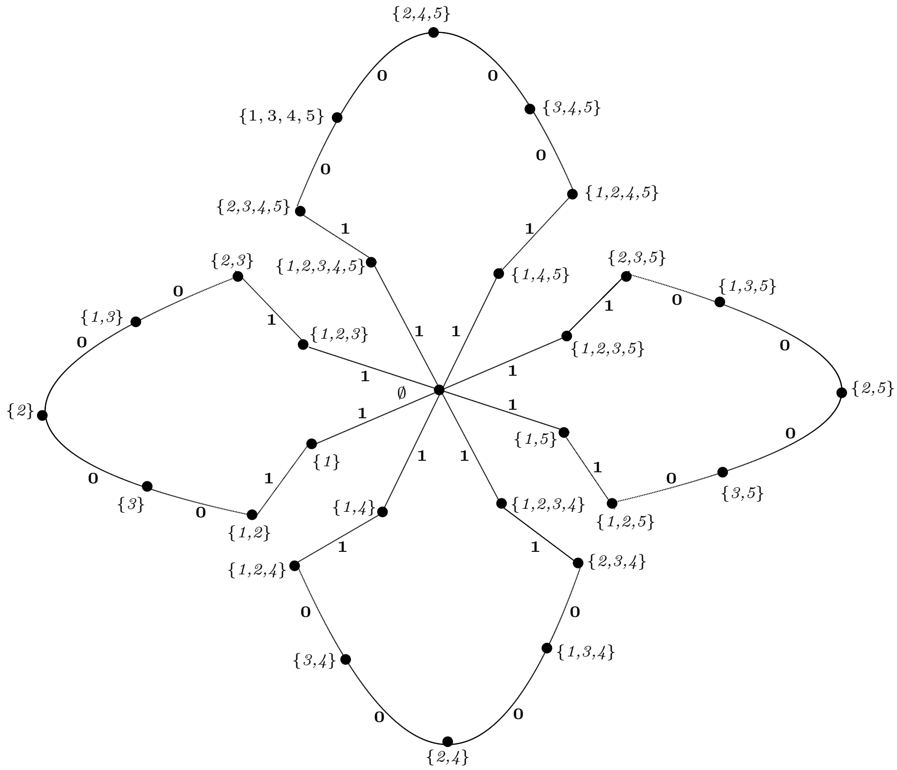


Figure 10: *Subset cordiality of one point union of cycles graph* $C_{2^3}^{(2^2)} = C_8^{(4)}$

Here, we have $e_f(0) = e_f(1) = 16$ *and* $|e_f(0) - e_f(1))| = 0$. *Thus* $C_8^{(4)}$ *is subset cordial.*

Definition 2.17. Let G be a graph and let $(P_p; G)$ be the graph obtained from m copies of G and the path $P_p : v_1v_2, \ldots, v_p$ by joining v_i with any vertex of the i^{th} copy of G by an edge, for $1 \leq i \leq p$.

Theorem 2.18. $(P_{2^n-1}; C_{2^n})$ is subset cordial, for $n \geq 3$.

Proof: Let $X = \{1, 2, \ldots, n, n+1, \ldots, 2n\}$. Note that the graph $(P_{2^n}; C_{2^n})$ has $2^n(2^n + 1)$ vertices and $2^n - 1 + 2^{2n} + 2^n$ edges. Label the vertices of P_{2^n-1} by the subsets of $\{1, 2, \ldots, n\}$ as in the Theorem 1.9 except the last subset $\{1, 2, \ldots, n\}$and hence $e_{f_{P_{2^n-1}}}(0) = e_{f_{P_{2^n-1}}}(1) = 2^n - 1$. Now add the element $n+1$ to each subset of $\{1, 2, \ldots, n\}$ and label these subsets as in the Theorem 1.10 to the first cycle attached to the first vertex of the path P_{2^n-1}. Here we join the first vertex of P_{2^n-1} to the vertex of first cycle labeled by the subset $\emptyset$. The corresponding edge gets the label 1.

Next, we add the element $n+2$ to each subset of $\{1, 2, \ldots, n\}$ and label the subsets to the second cycle attached to the second vertex of the path P_{2^n} and here join the second vertex of P_{2^n} to the vertex of the second cycle labeled by the subset which is not a subset of the subset labeled in the second vertex of P_{2^n}. We have 2^n subsets, so this is possible.

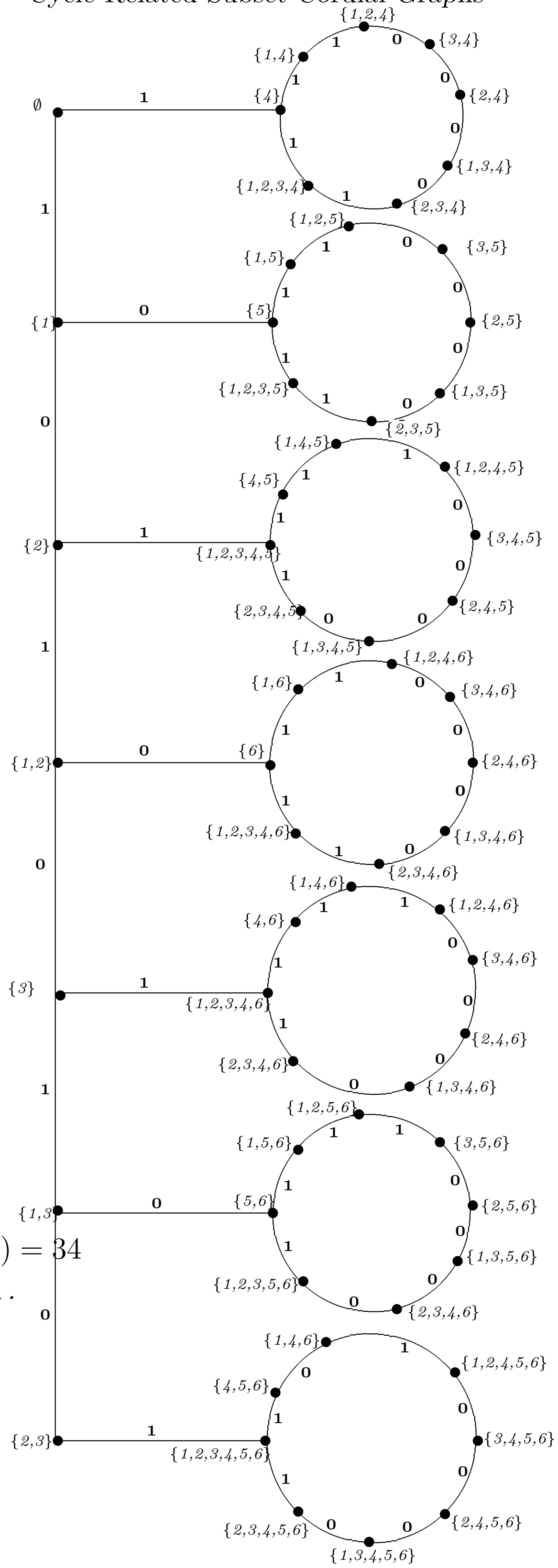
Hence the corresponding edge gets the label 0. Again we add the same element $n+2$ to each subset labeled to the vertices of the first cycle. We label these subsets to the third cycle attached to the third vertex of the path . Here, also we attach the third vertex of the path to the vertex of the cycle such that the one of the labels is a subset of the other and the corresponding edge gets the label 1.

Again using the element $n+3$, we can label the vertices of the next four cycles attached to the vertices of path. Continuing this process, we have such n elements and we can label all $1+2+2^2+\ldots+2^{n-1} = 2^n-1$ cycles attached to the pathP_{2^n-1}. By the construction, the attached edges get the labels 1 and 0 alternately. The labels obtained by the edges are shown below.

Components	$e_f(0)$	$e_f(1)$
1) For the path P_{2^n}(as per the Theorem)	$2^{n-1}-1$	$2^{n-1}-1$
2) For the 2^n cycles C_{2^n}(as per the Theorem)	2^{2n-1}	2^{2n-1}
3)Attached edges between path and the cycles	$2^{n-1}-1$	2^{n-1}
Total	$2^{2n-1}+2^n-2$	$2^{2n-1}+2^n-1$

Thus $|e_f(0)-e_f(1))|=1$ and hence $(P_{2^n-1};C_{2^n})$ is subset cordial. ■

Example 2.19. *Consider the graph* $(P_{2^3-1};C_{2^3}) = (P_7;C_8)$. *The subset cordiality is shown below. This graph has* 63 *vertices and* 69 *edges.*



Here $e_f(0) = 35$, $e_f(1) = 34$ *and* $|e_f(0) - e_f(1)| = 1$.

Figure 12: *Subset cordiality of* $(P_7 : C_8)$

Definition 2.20. Let G be a graph with vertex set $V(G)$. Let V' be the set of vertices with $|V'| = |V|$. For each vertex $a \in V$, we can associate a unique vertex $a' \in V'$. The duplicate graph of G, denoted by $D(G)$ has vertex set $V \cup V'$. If a and b are adjacent in G, then a', b and a, b' are adjacent in $D(G)$.

Definition 2.21. m_{C_n} is the graph obtained from m copies of C_n by attaching i^{th} copy of C_n to the vertex of $(i+1)^{th}$ copy of C_n has $m(n-1)+1$ vertices and mn edges.

Theorem 2.22. $2^m C_{2^n}$ is subset cordial for $n \geq 3$ and $m \geq 1$

Proof: Note that the graph $2^m C_{2^n}$ has $2^m(2^n - 1) + 1$ vertices and 2^{m+n} edges. Let $X = \{1, 2, \ldots, n, n+1, \ldots, n+m\}$. Assign the subsets of X to the vertices of $2^m C_{2^n}$

First, assign the 2^n subsets of $\{1, 2, \ldots, n\}$ to the vertices of first cycle in $2^m C_{2^n}$ as in the Theorem 2.22. Then the number of 1's and the number of 0's contributed to $e_f(1)$ and $e_f(0)$ respectively is 2^{n-1}each.

Now, we attach the second cycle to the first cycle by attaching the vertex labeled $\emptyset$ in the first cycle to the vertex of the second cycle. Thus the common vertex of the first cycle and the second cycle has the label $\emptyset$. Next, we construct $2^n - 1$ subsets by adding the element $n+1$ to each subset of $\{1, 2, \ldots, n\}$ except for $\emptyset$. Label these subsets to the $2^n - 1$ vertices of the second cycle. Note that already one vertex (which is common to the first and second cycle) has been labeled $\emptyset$. Then the number of 1's and 0's contributed to $e_f(1)$ and $e_f(0)$ respectively is $2^n - 1$ each. Then we construct $2^n - 1$ subsets by adding the element $n+2$ to each subset of $\{1, 2, \ldots, n\}$ except for $\emptyset$. We assign these $2^n - 1$ subsets to the third cycle, as in the first cycle. Next, we attach the third cycle to the fourth cycle by attaching the vertex labeled $\{n+2\}$ in the third cycle to the vertex in the fourth cycle. Next, we construct $2^n - 1$ subsets by adding the same element $n+2$ to each subset labeled in the second cycle, except for $\emptyset$. Then label these subsets to the vertices of the fourth cycle.

Continuing this process, we observe the following.

i)Using the subsets of $\{1, 2, \ldots, n\}$, we labeled the first cycle.

ii)Adding the element $n+1$ to the subsets of $\{1, 2, \ldots, n\}$, we labeled the second cycle.

iii) Adding the element $n+2$ to the subsets of $\{1, 2, \ldots, n, n+1\}$, we labeled the third and fourth cycles.

Continuing this process, adding the element $n+m$ to the subsets of $\{1, 2, \ldots, n, n+1, \ldots, n+m-1\}$, we have labeled the last $2^m - 1$ cycles as $2^m C_{2^n}$. Note that the number of cycles in $2^m C_{2^n}$ is checked by calculating $1+2^0+2^1+2^2+\ldots+2^{m-1} = 1+\frac{2^m-1}{2-1} = 2^m$. In each cycle, we also note that the number of 1's and the number of 0's contributed to $e_f(1)$ and $e_f(0)$ respectively is $2^n - 1$ each.

Thus $e_f(0) = 2^{m+n-1} = e_f(1)$ and $|e_f(0) - e_f(1)| = 0$. Hence $2^m C_{2^n}$ is subset cordial for $n \geq 3$ and $m \geq 1$. ∎

The Theorem 2.22 is illustrated in the following example.

Example 2.23. *Let $X = \{1, 2, 3, 4, 5\}$. The subset cordiality of $2^2 C_{2^3}$ is given below. Here we note that the subsets of $\{1, 2, 3\}$ are labeled in the first cycle. Using the element 4, we have labeled the second cycle as in the theorem 2.22 and using the element 5, we have labeled the third and fourth cycles.*

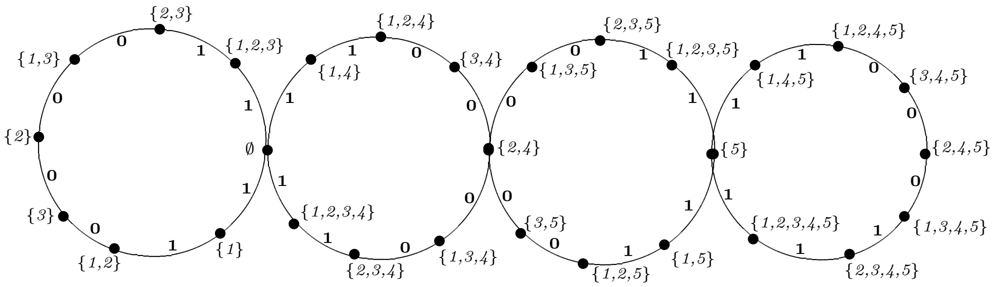


Figure 13: *A subset cordial labeling of graph $2^2 C_{2^3}$*

We have $e_f(0) = 16 = e_f(1)$ and $|e_f(0) - e_f(1)| = 0$.

Thus the graph $2^2 C_{2^3}$ is subset cordial.

Theorem 2.24. Wheel with two centers $W_p^{(2)} = \overline{K}_2 + C_{p-2}$ is subset cordial.

Proof: Let $X = \{1, 2, \ldots, n\}$. We have $W_p^{(2)} = \overline{K}_2 + C_{p-2}$. Let $v_1, v_2, \ldots, v_{p-2}$ be the vertices of C_{p-2} and v_{p-1}. Let v_p be the central vertices of $W_p^{(2)}$. Note that $q(W_p^{(2)}) = 3(p-2)$. Take $p = 2^n$. Then $q = 3(2^n - 2)$. Now we label the vertices of $W_p^{(2)}$ by combining the labeling pattern for the star $K_{1,p-1}$(two times) and the cycle described in the Theorem 1.12 and 1.10 respectively.

First, we label the central vertex v_p by the subset $\{1, 2, \ldots, n-1\}$ as in the Theorem 1.12 and label another central vertex v_{p-1} by the singleton subset $\{1\}$. We also label the vertices v_1,v_2,…,v_{p-2} as in the Theorem 1.10 except the labels$\{1, 2, \ldots, n-1\}$ and $\{1\}$ which were labeled to v_p and v_{p-1} respectively.

The spokes of the wheel $W_p^{(2)}$ with respect to the center v_p contribute $2^{n-1}-1$ each to $e_f(1)$ and $e_f(0)$ as in the Theorem 1.12. The spokes of the wheel with respect to the center v_{p-1} contribute $2^{n-1}-1$ each to $e_f(1)$ and $e_f(0)$ as in the above Note (2.36). From the Theorem 1.10, it follows that the rim of the wheel W_p contributes $2^{n-1}-1$ to each $e_f(1)$ and $e_f(0)$. Hence $e_f(0)$= $e_f(1)$=$3(2^{n-1}-1)$. Thus $W_p^{(2)}$ is subset cordial. ∎

Example 2.25. Let $X=\{1,2,3,4\}$. Consider the wheel with two centres,$W_p^{(2)}$

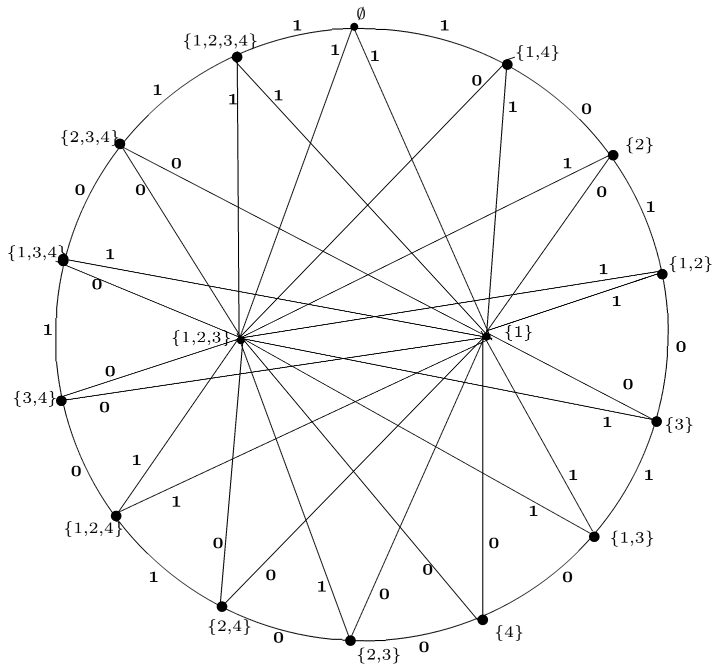


Figure 14: *The subset cordiality of wheel with two centres, $W_p^{(2)}$* ∎

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