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Some Topological Indices of Strong Product of Graphs

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Abstract

The aim of this paper is to derive explicit formulae for several important topological indices, including the first Zagreb index, second Zagreb index, hyper Zagreb index, second hyper Zagreb index, and Gourava index, for the strong product of two graphs. These formulae are expressed in terms of the structural properties of the constituent graphs, such as their degree sequences, as well as the interactions between their vertices and edges under the strong product operation. In particular, we investigate the strong product $G_1 \boxtimes G_2$ of two graphs G_1 and G_2 and establish explicit expressions for the aforementioned topological indices.

Key words: First Zagreb index, second Zagreb index, hyper Zagreb index, Second hyper Zagreb index, Second Gourava index.

2020 Mathematics Subject Classification : $05C07, 05C10, 05C76$

1 Introduction

All graphs considered here are simple, connected and finite. Let $V(G), E(G), d_G(v)$ and $d_G(u, v)$ denote the vertex set, the edge set, the degree of a vertex and the distance between the vertices u and v of a graph G respectively. A graph with n vertices and m edges is called a (n, m) graph. First we present the definitions and notations which are required throughout this paper.

A topological index of a graph G is a real number which is invariant under automorphism of G and does not depend on the labeling or pictorial representation of a graph. The Wiener index [8] is the first and most studied topological indices and it is defined as

$$W(G) = \frac{1}{2} \sum_{\{u,v\} \subseteq V(G)} d_G(u, v).$$

Gutman et.al., [3] introduced the first and second Zagreb indices of a graph G as follows:

$$M_1(G) = \sum_{uv \in E(G)} (d_G(u) + d_G(v)) = \sum_{u \in V(G)} d_G^2(u) \text{ and } M_2(G) = \sum_{uv \in E(G)} d_G(u) d_G(v)$$

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Shirdel et.al [7] defined the hyper-Zagreb index as

$$HM(G) = \sum_{uv \in E(G)} [d_G(u) + d_G(v)]^2$$

and presented the exact formulae for the Hyper-Zagreb index of some well-known graphs.

Farahani et.al [1] defined the second hyper-Zagreb index as

$$HM_2(G) = \sum_{uv \in E(G)} d_G^2(u) d_G^2(v)$$

A forgotten topological index of F-index [2] is defined for a graph G as

$$F(G) = \sum_{v \in V(G)} d_G^3(v) = \sum_{uv \in E(G)} [d_G^2(u) + d_G^2(v)]$$

Here we introduce a second forgotten topological index F_2 which is defined for a graph G as

$$F_2(G) = \sum_{v \in V(G)} d_G^4(v)$$

V. R. Kulli [5] introduced the first and second Gourava indices, defined as

$$GO_1(G) = \sum_{uv \in E(G)} [d_G(u) + d_G(v) + d_G(u) d_G(v)], \quad (1)$$

$$GO_2(G) = \sum_{uv \in E(G)} d_G(u) d_G(v) [d_G(u) + d_G(v)]. \quad (2)$$

The strong product of the graphs G_1 and G_2 denoted by $G_1 \boxtimes G_2$ is the graph with vertex set $V(G_1) \times V(G_2)$ and two vertices (u_1, u_2) and (v_1, v_2) are adjacent whenever (i) $u_1 = v_1$ and $u_2 v_2 \in E(G_2)$ or (ii) $u_2 = v_2$ and $u_1 v_1 \in E(G_1)$ or $u_1 v_1 \in E(G_1)$ and $u_2 v_2 \in E(G_2)$.

We begin this paper with the following Lemma.

Lemma 1.1. [4] $d_{G_1 \boxtimes G_2}(u_i, v_j) = d_{G_1}(u_i) + d_{G_2}(v_j) + d_{G_1}(u_i) d_{G_2}(v_j)$, *where* (u_i, v_j) *of* $V(G_1 \boxtimes G_2)$

Remark 1.2. [6] For a graph G, let $A(G) = \{(x, y) \in V(G) \times V(G) \mid x \text{ and } y \text{ are adjacent in } G\}$ The summation $\sum_{(x,y) \in A(G)}$ runs over the ordered pairs of $A(G)$. For simplicity, we write the summation $\sum_{(x,y) \in A(G)}$ as $\sum_{xy \in G}$.

Lemma 1.3. *(a)* $\sum_{xy \in G} 1 = 2e(G)$

(b) $\sum_{xy \in G} d_G(x) = M_1(G)$

(c) $\sum_{xy \in G} d_G(x) d_G(y) = 2M_2(G)$ *and (d)* $\sum_{xy \in G} d_G^2(x) = F(G)$

Lemma 1.4. *Let G be a graph. Then* $\sum_{xy\in G}[d_G(x)+d_G(y)]^2 = 2HM(G)$.

Proof: $\sum_{xy\in G}[d_G(x)+d_G(y)]^2 = 2\sum_{xy\in E(G)}[d_G(x)+d_G(y)]^2 = 2HM(G)$ ■

Lemma 1.5. *Let G be a graph. Then* $\sum_{xy\in G} d_G^2(x)d_G^2(y) = 2HM_2(G)$.

Proof: $\sum_{xy\in G} d_G^2(x)d_G^2(y) = 2\sum_{xy\in E(G)} d_G^2(x)d_G^2(y) = 2HM_2(G)$ ■

Lemma 1.6. *Let G be a graph. Then* $\sum_{xy\in G} d_G(x)d_G(y)[d_G(x)+d_G(y)] = 2GO_2(G)$.

Proof: $\sum_{xy\in G} d_G(x)d_G(y)[d_G(x)+d_G(y)] = 2\sum_{xy\in E(G)} d_G(x)d_G(y)[d_G(x)+d_G(y)] = 2GO_2(G)$ ■

2 Main Results

2.1 The First Zagreb Index of Strong Product of Graphs

In this section, we find the exact value of the first Zagreb index of strong product of graphs.

Theorem 2.1. *Let $G_i, i = 1, 2$ be a (n_i, m_i) graph. Then*

$$M_1(G_1 \boxtimes G_2) = M_1(G_1)M_1(G_2) + (4m_2+n_2)M_1(G_1) + (4m_1+n_1)M_1(G_2) + 8m_1m_2$$

Proof:

$$\begin{aligned}
2M_1(G_1 \boxtimes G_2) &= \sum_{(u,x)(v,y)\in G_1\boxtimes G_2} [d_{G_1\boxtimes G_2}(u,x) + d_{G_1\boxtimes G_2}(v,y)] \\
&= \sum_{u\in V(G_1)}\sum_{xy\in G_2}[[d_{G_1}(u)+d_{G_2}(x)+d_{G_1}(u)d_{G_2}(x)] \\
&\quad +[d_{G_1}(u)+d_{G_2}(y)+d_{G_1}(u)d_{G_2}(y)]] \\
&\quad + \sum_{x\in V(G_2)}\sum_{uv\in G_1}[[d_{G_1}(u)+d_{G_2}(x)+d_{G_1}(u)d_{G_2}(x)] \\
&\quad +[d_{G_1}(v)+d_{G_2}(x)+d_{G_1}(v)d_{G_2}(x)]] \\
&\quad + \sum_{xy\in G_2}\sum_{uv\in G_1}[[d_{G_1}(u)+d_{G_2}(x)+d_{G_1}(u)d_{G_2}(x)] \\
&\quad +[d_{G_1}(v)+d_{G_2}(y)+d_{G_1}(v)d_{G_2}(y)]] \\
&= S_1 + S_2 + S_3
\end{aligned}$$

where S_1, S_2 and S_3 are the sums of the above order.

$$\begin{aligned}
S_1 &= \sum_{u\in V(G_1)}\sum_{xy\in G_2}[[d_{G_1}(u)+d_{G_2}(x)+d_{G_1}(u)d_{G_2}(x)] \\
&\quad +[d_{G_1}(u)+d_{G_2}(y)+d_{G_1}(u)d_{G_2}(y)]]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{u\in V(G_1)}\sum_{xy\in G_2}[2d_{G_1}(u)+d_{G_1}(u)(d_{G_2}(x)+d_{G_2}(x))+(d_{G_2}(x)+d_{G_2}(x))]\\
&= 8m_1m_2+4m_1M_1(G_2)+2n_1M_1(G_2)\\
S_2 &= \sum_{x\in V(G_2)}\sum_{uv\in G_1}[[d_{G_1}(u)+d_{G_2}(x)+d_{G_1}(u)d_{G_2}(x)]\\
&\quad +[d_{G_1}(v)+d_{G_2}(x)+d_{G_1}(v)d_{G_2}(x)]]\\
&= \sum_{x\in V(G_2)}\sum_{uv\in G_1}[(d_{G_1}(u)+d_{G_1}(v))+2d_{G_2}(x)+d_{G_2}(x)(d_{G_1}(u)+d_{G_1}(v))]\\
&= 2n_2M_1(G_1)+8m_1m_2+4m_2M_1(G_1)\\
S_3 &= \sum_{xy\in G_2}\sum_{uv\in G_1}[[d_{G_1}(u)+d_{G_2}(x)+d_{G_1}(u)d_{G_2}(x)]\\
&\quad +[d_{G_1}(v)+d_{G_2}(y)+d_{G_1}(v)d_{G_2}(y)]]\\
&= \sum_{xy\in G_2}\sum_{uv\in G_1}[(d_{G_1}(u)+d_{G_1}(v))+(d_{G_2}(x)+d_{G_2}(y))\\
&\quad +(d_{G_1}(u)d_{G_2}(x)+d_{G_1}(v)d_{G_2}(y))]\\
&= 4m_2M_1(G_1)+4m_1M_1(G_2)+2M_1(G_1)M_1(G_2)
\end{aligned}
$$

Adding S_1, S_2 and S_3 we get the desired result. ∎

2.2 The Second Zagreb Index of Strong Product of Graphs

In this section, we find the exact value of the second Zagreb index of strong product of graphs.

Theorem 2.2. *Let $G_i, i = 1, 2$ be a (n_i, m_i) graph. Then*

$$
\begin{aligned}
M_2(G_1\boxtimes G_2) &= M_2(G_2)[n_1+6m_1+3M_1(G_1)+M_2(G_1)]+M_2(G_1)[n_2+6m_2\\
&\quad +3M_1(G_2)]+M_1(G_2)[3m_1+2M_1(G_1)]+M_1(G_1)[3m_2+M_1(G_2)]
\end{aligned}
$$

Proof:

$$
\begin{aligned}
2M_2(G_1\boxtimes G_2) &= \sum_{(u,x)(v,y)\in G_1\boxtimes G_2} d_{G_1\boxtimes G_2}(u,x)d_{G_1\boxtimes G_2}(v,y)\\
&= \sum_{u\in V(G_1)}\sum_{xy\in G_2}[d_{G_1}(u)+d_{G_2}(x)+d_{G_1}(u)d_{G_2}(x)]\\
&\quad [d_{G_1}(u)+d_{G_2}(y)+d_{G_1}(u)d_{G_2}(y)]\\
&\quad +\sum_{x\in V(G_2)}\sum_{uv\in G_1}[d_{G_1}(u)+d_{G_2}(x)+d_{G_1}(u)d_{G_2}(x)]\\
&\quad [d_{G_1}(v)+d_{G_2}(x)+d_{G_1}(v)d_{G_2}(x)]\\
&\quad +\sum_{xy\in G_2}\sum_{uv\in G_1}[d_{G_1}(u)+d_{G_2}(x)+d_{G_1}(u)d_{G_2}(x)]\\
&\quad [d_{G_1}(v)+d_{G_2}(y)+d_{G_1}(v)d_{G_2}(y)]\\
&= S_1+S_2+S_3
\end{aligned}
$$

where S_1, S_2 and S_3 are the sums of the above order.

$$\begin{aligned}
S_1 &= \sum_{u\in V(G_1)}\sum_{xy\in G_2}[d_{G_1}(u)+d_{G_2}(x)+d_{G_1}(u)d_{G_2}(x)]\\
&\quad [d_{G_1}(u)+d_{G_2}(y)+d_{G_1}(u)d_{G_2}(y)\\
&= \sum_{u\in V(G_1)}\sum_{xy\in G_2}[d^2_{G_1}(u)d_{G_2}(x)d_{G_2}(y)+2d_{G_1}(u)d_{G_2}(x)d_{G_2}(y)+d_{G_2}(x)d_{G_2}(y)]\\
&\quad d^2_{G_1}(u)(d_{G_2}(x)+d_{G_2}(y))+d_{G_1}(u)(d_{G_2}(x)+d_{G_2}(y))+d^2_{G_1}(u)]\\
&= 2M_1(G_1)M_2(G_2)+8m_1M_2(G_2)+2n_1M_2(G_2)+2M_1(G_1)M_1(G_2)\\
&\quad +4m_1M_1(G_2)+2m_2M_1(G_1)\\
&= 2M_2(G_2)[n_1+4m_1+M_1(G_1)]+2M_1(G_2)[2m_1+M_1(G_1)]+2m_2M_1(G_1)\\
S_2 &= \sum_{x\in V(G_2)}\sum_{uv\in G_1}[d_{G_1}(u)+d_{G_2}(x)+d_{G_1}(u)d_{G_2}(x)]\\
&\quad [d_{G_1}(v)+d_{G_2}(x)+d_{G_1}(v)d_{G_2}(x)]\\
&= \sum_{x\in V(G_2)}\sum_{uv\in G_1}[d^2_{G_2}(x)d_{G_1}(u)d_{G_1}(v)+2d_{G_2}(x)d_{G_1}(u)d_{G_1}(v)+d_{G_1}(u)d_{G_1}(v)\\
&\quad +d^2_{G_2}(x)(d_{G_1}(u)+d_{G_1}(v))+d_{G_2}(x)(d_{G_1}(u)+d_{G_1}(v))+d^2_{G_2}(x)]\\
&= 2M_2(G_1)M_1(G_2)+8m_2M_2(G_1)+2n_2M_2(G_1)+2M_1(G_1)M_1(G_2)\\
&\quad +4m_2M_1(G_1)+2m_1M_1(G_2)\\
&= 2M_2(G_1)[n_2+4m_2+M_1(G_2)]+2M_1(G_1)[2m_2+M_1(G_2)]+2m_1M_1(G_2)\\
S_3 &= \sum_{xy\in G_2}\sum_{uv\in G_1}[d_{G_1}(u)d_{G_2}(x)+d_{G_1}(u)d_{G_2}(x)][d_{G_1}(v)+d_{G_2}(y)+d_{G_1}(v)d_{G_2}(y)]\\
&= \sum_{xy\in G_2}\sum_{uv\in G_1}[d_{G_1}(u)d_{G_1}(v)+d_{G_2}(x)d_{G_2}(y)+d_{G_1}(u)d_{G_1}(v)(d_{G_2}(x)+d_{G_2}(y))\\
&\quad +d_{G_2}(x)d_{G_2}(y)(d_{G_1}(u)+d_{G_1}(v))+d_{G_1}(u)d_{G_2}(y)+d_{G_1}(v)d_{G_2}(x)\\
&\quad +d_{G_1}(u)d_{G_1}(v)d_{G_2}(x)d_{G_2}(y)]\\
&= 4m_2M_2(G_1)+4m_1M_2(G_2)+4M_2(G_1)M_1(G_2)+4M_1(G_1)M_2(G_2)\\
&\quad +2M_1(G_1)M_1(G_2)+4M_2(G_1)M_2(G_2)\\
&= 4M_2(G_2)[m_1+M_1(G_1)+m_2(G_1)]+4M_2(G_1)[m_2+M_1(G_2)]+2M_1(G_1)M_1(G_2)
\end{aligned}$$

Adding the above inequalities we get the desired result. ■

2.3 The Hyper Zagreb Index of Strong Product of Graphs

In this section, we find the exact value of the hyper Zagreb index of strong product of graphs.

Theorem 2.3. *Let $G_i, i = 1, 2$ be a (n_i, m_i) graph. Then*

$$HM(G_1 \boxtimes G_2) = HM(G_2)[n_1+6m_1+M_1(G_1)]+HM(G_1)[n_2+4m_2+M_1(G_2)]$$

$$+2M_1(G_2)[6m_1 + M_2(G_1) + F(G_1)] + 2M_1(G_1)[6m_2 + M_2(G_2) + F(G_2)]$$
$$+12M_1(G_1)M_1(G_2) + 4M_2(G_1)M_2(G_2) + F(G_1)F(G_2)$$

Proof:

$$\begin{aligned}
2HM(G_1 \boxtimes G_2) &= \sum_{(u,x)(v,y)\in G_1\boxtimes G_2} [d_{G_1\boxtimes G_2}(u,x) + d_{G_1\boxtimes G_2}(v,y)]^2 \\
&= \sum_{u\in V(G_1)} \sum_{xy\in G_2} [d_{G_1}(u) + d_{G_2}(x) + d_{G_1}(u)d_{G_2}(x) + d_{G_1}(u) \\
&\quad +d_{G_2}(y) + d_{G_1}(u)d_{G_2}(y)]^2 \\
&\quad + \sum_{x\in V(G_2)} \sum_{uv\in G_1} [d_{G_1}(u) + d_{G_2}(x) + d_{G_1}(u)d_{G_2}(x) \\
&\quad +d_{G_1}(v) + d_{G_2}(x) + d_{G_1}(v)d_{G_2}(x)]^2 \\
&\quad + \sum_{xy\in G_2} \sum_{uv\in G_1} [d_{G_1}(u) + d_{G_2}(x) + d_{G_1}(u)d_{G_2}(x) + d_{G_1}(v) \\
&\quad +d_{G_2}(y) + d_{G_1}(v)d_{G_2}(y)]^2 \\
&= S_1 + S_2 + S_3
\end{aligned}$$

where S_1, S_2 and S_3 are the sums of the above order.

$$\begin{aligned}
S_1 &= \sum_{u\in V(G_1)} \sum_{xy\in G_2} [d_{G_1}(u) + d_{G_2}(x) + d_{G_1}(u)d_{G_2}(x) + d_{G_1}(u) + d_{G_2}(y) + d_{G_1}(u)d_{G_2}(y)]^2 \\
&= \sum_{u\in V(G_1)} \sum_{xy\in G_2} [2d_{G_1}(u) + d_{G_2}(x) + d_{G_2}(y) + d_{G_1}(u)(d_{G_2}(x) + d_{G_2}(y))]^2 \\
&= \sum_{u\in V(G_1)} \sum_{xy\in G_2} [4d^2_{G_1}(u) + (d_{G_2}(x) + d_{G_2}(y))^2 + d^2_{G_1}(u)(d_{G_2}(x) + d_{G_2}(y))^2 \\
&\quad +4d_{G_1}(u)(d_{G_2}(x) + d_{G_2}(y)) + 4d^2_{G_1}(u)(d_2(x) + d_{G_2}(y)) \\
&\quad +4d_{G_1}(u)(d_{G_2}(x) + d_{G_2}(y))^2] \\
&= 8m_2M_1(G_1) + 2n_1HM(G_2) + 2M_1(G_1)HM(G_2) + 16m_1M_1(G_2) \\
&\quad +8M_1(G_1)M_1(G_2) + 8m_1HM(G_2) \\
&= 2HM(G_2)[n_1 + 4m_1 + M_1(G_1)] + 8M_1(G_2)[2m_1 + M_1(G_1)] + 8m_2M_1(G_1) \\
S_2 &= \sum_{x\in V(G_2)} \sum_{uv\in G_1} [d_{G_1}(u) + d_{G_2}(x) + d_{G_1}(u)d_{G_2}(x) + d_{G_1}(v) + d_{G_2}(x) + d_{G_1}(v)d_{G_2}(x)]^2 \\
&= \sum_{x\in V(G_2)} \sum_{uv\in G_1} [2d_{G_2}(x) + d_{G_1}(u) + d_{G_1}(v) + d_{G_2}(x)(d_{G_1}(u) + d_{G_1}(v))]^2 \\
&= \sum_{x\in V(G_2)} \sum_{uv\in G_1} [4d^2_{G_2}(x) + (d_{G_1}(u) + d_{G_1}(v))^2 + d^2_{G_2}(x)(d_{G_1}(u) + d_{G_1}(v))^2 \\
&\quad +4d_{G_2}(x)(d_{G_1}(u) + d_{G_1}(v)) + 4d^2_{G_2}(x)(d_{G_1}(u) + d_{G_1}(v)) \\
&\quad +4d_{G_2}(x)(d_{G_1}(u) + d_{G_1}(v))^2] \\
&= 8m_1M_1(G_2) + 2n_2HM(G_1) + 2M_1(G_2)HM(G_1) + 16m_2M_1(G_1)
\end{aligned}$$

$$
\begin{aligned}
&+8M_1(G_2)M_1(G_1) + 8m_2HM(G_1)\\
&= 2HM(G_1)[n_2 + 4m_2 + M_1(G_2)] + 8M_1(G_1)[2m_2 + M_1(G_2)] + 8m_1M_1(G_2)\\
S_3 &= \sum_{xy\in G_2}\sum_{uv\in G_1}[d_{G_1}(u) + d_{G_2}(x) + d_{G_1}(u)d_{G_2}(x) + d_{G_1}(v) + d_{G_2}(y) + d_{G_1}(v)d_{G_2}(y)]^2\\
&= \sum_{xy\in G_2}\sum_{uv\in G_1}[(d_{G_1}(u) + d_{G_1}(v)) + (d_{G_2}(x) + d_{G_2}(y)) + d_{G_1}(u)d_{G_2}(x) + d_{G_1}(v)d_{G_2}(y)]\\
&= \sum_{xy\in G_2}\sum_{uv\in G_1}[(d_{G_1}(u) + d_{G_1}(v))^2 + (d_{G_2}(x) + d_{G_2}(y))^2 + d^2_{G_1}(u)d^2_{G_2}(x)\\
&\quad +d^2_{G_1}(v)d^2_{G_2}(y)] + 2(d_{G_1}(u) + d_{G_1}(v))(d_{G_2}(x) + d_{G_2}(y)) + 2d^2_{G_1}(u)d_{G_2}(x)\\
&\quad +2d_{G_1}(u)d_{G_1}(v)d_{G_2}(x) + 2d_{G_1}(u)d_{G_1}(v)d_{G_2}(y) + 2d^2_{G_1}(v)d_{G_2}(y)\\
&\quad +2d_{G_1}(u)d^2_{G_2}(x) + 2d_{G_1}(u)d_{G_2}(x)d_{G_2}(y) + 2d_{G_1}(v)(d_{G_2}(x)d_{G_2}(y)\\
&\quad +2d_{G_1}(v)d^2_{G_2}(y) + 2d_{G_1}(u)d_{G_2}(x)d_{G_1}(v)d_{G_2}(y)]\\
&= 4m_2HM(G_1) + 4m_1HM(G_2) + F(G_1)F(G_2) + F(G_1)F(G_2) + 8M_1(G_1)M_1(G_2)\\
&\quad +2F(G_1)M_1(G_2) + 4M_2(G_1)M_1(G_2) + 4M_2(G_1)M_1(G_2) + 2F(G_1)M_1(G_2)\\
&\quad +4M_1(G_1)M_2(G_2) + 4M_1(G_1)M_2(G_2) + 2M_1(G_1)F(G_2) + 2M_1(G_1)F(G_2)\\
&\quad +8M_2(G_1)M_2(G_2)\\
&= 4m_2HM(G_1) + 4m_1HM(G_2) + 2F(G_1)F(G_2) + 8M_1(G_1)M_1(G_2)\\
&\quad +8M_2(G_1)M_2(G_2) + 8M_1(G_1)M_2(G_2) + 8M_2(G_1)M_1(G_2) + 4F(G_1)M_1(G_2)\\
&\quad +4M_1(G_1)F(G_2)
\end{aligned}
$$

Adding the above inequalities we get the desired result. ∎

2.4 The Second Hyper Zagreb Index of Strong Product of graphs

In this section, we find the exact value of the second hyper Zagreb index of strong product of graphs.

Theorem 2.4. *Let $G_i, i = 1, 2$ be a (n_i, m_i) graph. Then*

$$
\begin{aligned}
HM_2(G_1 \boxtimes G_2) &= 6GO_2(G_1)GO_2(G_2) + 2HM_2(G_1)HM_2(G_2) + F(G_1)F(G_2)\\
&\quad +8M_2(G_1)M_2(G_2)\\
&\quad +2GO_2(G_2)[2m_1 + 4M_1(G_1) + 4M_2(G_1) + 4F(G_1) + 3F_2(G_1)\\
&\quad +2HM_2(G_1)] + 2GO_2(G_1)[2m_2 + 4M_1(G_2) + 4M_2(G_2)\\
&\quad +4F(G_2) + 3F_2(G_2) + 2HM_2(G_2)]\\
&\quad +2HM_2(G_2)[n_1 + 9m_1 + 10M_1(G_1) + 8M_2(G_1) + 4F(G_1) + F_2(G_1)]\\
&\quad +2HM_2(G_1)[n_2 + 9m_2 + 10M_1(G_2) + 8M_2(G_2) + 4F(G_2) + F_2(G_2)]\\
&\quad +F(G_2)[M_1(G_1) + 2F(G_1) + F_2(G_1)]
\end{aligned}
$$

$$
\begin{aligned}
&+4M_2(G_2)[M_1(G_1)+2F(G_1)+F_2(G_1)]\\
&+2M_1(G_2)[F(G_1)+F_2(G_1)+m_2F_2(G_1)]\\
&+F(G_1)[M_1(G_2)+2F(G_2)+F_2(G_2)]\\
&+4M_2(G_1)[M_1(G_2)+2F(G_2)+F_2(G_2)]\\
&+2M_1(G_1)[F(G_2)+F_2(G_2)+m_1F_2(G_2)]
\end{aligned}
$$

Proof:

$$
\begin{aligned}
2HM_2(G_1\boxtimes G_2) &= \sum_{(u,x)(v,y)\in G_1\boxtimes G_2} d^2_{G_1\boxtimes G_2}(u,x)d^2_{G_1\boxtimes G_2}(v,y)\\
&= \sum_{u\in V(G_1)}\sum_{xy\in G_2}[d_{G_1}(u)+d_{G_2}(x)+d_{G_1}(u)d_{G_2}(x)]^2\\
&\quad [d_{G_1}(u)+d_{G_2}(y)+d_{G_1}(u)d_{G_2}(y)]^2\\
&\quad +\sum_{x\in V(G_2)}\sum_{uv\in G_1}[d_{G_1}(u)+d_{G_2}(x)+d_{G_1}(u)d_{G_2}(x)]^2\\
&\quad [d_{G_1}(v)+d_{G_2}(x)+d_{G_1}(v)d_{G_2}(x)]^2\\
&\quad +\sum_{xy\in G_2}\sum_{uv\in G_1}[d_{G_1}(u)+d_{G_2}(x)+d_{G_1}(u)d_{G_2}(x)]^2\\
&\quad [d_{G_1}(v)+d_{G_2}(y)+d_{G_1}(v)d_{G_2}(y)]^2\\
&= S_1+S_2+S_3
\end{aligned}
$$

where S_1, S_2 and S_3 are the sums of the above order.

$$
\begin{aligned}
S_1 &= \sum_{u\in V(G_1)}\sum_{xy\in G_2}[d_{G_1}(u)+d_{G_2}(x)+d_{G_1}(u)d_{G_2}(x)]^2[d_{G_1}(u)+d_{G_2}(y)+d_{G_1}(u)d_{G_2}(y)]^2\\
&= \sum_{u\in V(G_1)}\sum_{xy\in G_2}[d^2_{G_1}(u)+d^2_{G_2}(x)+d^2_{G_1}(u)d^2_{G_2}(x)+2d_{G_1}(u)d_{G_2}(x)\\
&\quad +2d^2_{G_1}(u)d_{G_2}(x)+d_{G_1}(u)d^2_{G_2}(x)][d^2_{G_1}(u)+d^2_{G_2}(y)+d^2_{G_1}(u)d^2_{G_2}(y)\\
&\quad +2d_{G_1}(u)d_{G_2}(y)+2d^2_{G_1}(u)d_{G_2}(y)+d_{G_1}(u)d^2_{G_2}(y)]\\
&= \sum_{u\in V(G_1)}\sum_{xy\in G_2}[6d^4_{G_1}(u)d_{G_2}(x)d_{G_2}(y)(d_2(x)+d_{G_2}(y))\\
&\quad +6d^3_{G_1}(u)d_{G_2}(x)d_{G_2}(y)(d_2(x)+d_{G_2}(y))+6d^2_{G_1}(u)d_{G_2}(x)d_{G_2}(y)(d_2(x)+d_{G_2}(y))\\
&\quad +2d^2_{G_1}(u)d_{G_2}(x)d_{G_2}(y)(d_2(x)+d_{G_2}(y))+d^4_{G_1}(u)d^2_{G_2}(x)d^2_{G_2}(y)\\
&\quad +4d^3_{G_1}(u)d^2_{G_2}(x)d^2_{G_2}(y)+6d^2_{G_1}(u)d^2_{G_2}(x)d^2_{G_2}(y)+4d_{G_1}(u)d^2_{G_2}(x)d^2_{G_2}(y)\\
&\quad +d^4_{G_1}(u)(d^2_{G_2}(x)+d^2_{G_2}(y))+2d^4_{G_1}(u)(d^2_{G_2}(x)+d^2_{G_2}(y))+d^4_{G_1}(u)\\
&\quad +d^2_{G_1}(u)(d^2_{G_2}(x)+d^2_{G_2}(y))+4d^4_{G_1}(u)d_{G_2}(x)d_{G_2}(y)+8d^3_{G_1}(u)d_{G_2}(x)d_{G_2}(y)\\
&\quad +4d^2_{G_1}(u)d_{G_2}(x)d_{G_2}(y)+2d^4_{G_1}(u)(d_{G_2}(x)+d_{G_2}(y))+2d^3_{G_1}(u)(d_{G_2}(x)+d_{G_2}(y))]\\
&= 4F_2(G_1)GO_2(G_2)+12F(G_1)GO_2(G_2)+12M_1(G_1)GO_2(G_2)+8m_1GO_2(G_2)
\end{aligned}
$$

$$
\begin{aligned}
&+12M_1(G_1)HM_2(G_2)+16m_1HM_2(G_2)+2F_2(G_1)HM_2(G_2)+8F(G_1)HM_2(G_2)\\
&+2n_1HM_2(G_2)+2F_2(G_1)F(G_2)+4F(G_1)F(G_2)+2M_1(G_1)F(G_2)\\
&+8F_2(G_1)M_2(G_2)+16F(G_1)M_2(G_2)+8M_1(G_1)M_2(G_2)+4F_2(G_1)M_1(G_2)\\
&+4F(G_1)M_1(G_2)+2m_2F_2(G_1)\\
=\;& 4GO_2(G_2)[2m_1+3M_1(G_1)+3F(G_1)+3F_2(G_1)]\\
&+2HM_2(G_2)[n_1+8m_1+6M_1(G_1)+4F(G_1)+F_2(G-1)]\\
&+2F(G_2)[M_1(G_1)+2F(G_1)+F_2(G_1)]+8M_2(G_2)[M_1(G_1)+2F(G_1)+F_2(G_1)]\\
&+4M_1(G_2)[F(G_1)+F_2(G_1)+2m_2F_2(G_1)]
\end{aligned}
$$

$$
\begin{aligned}
S_2 =\;& \sum_{x\in V(G_2)}\sum_{uv\in G_1}[d_{G_1}(u)+d_{G_2}(x)+d_{G_1}(u)d_{G_2}(x)]^2[d_{G_1}(v)+d_{G_2}(x)+d_{G_1}(v)d_{G_2}(x)]^2\\
=\;& \sum_{u\in V(G_1)}\sum_{xy\in G_2}[d^2_{G_1}(u)+d^2_{G_2}(x)+d^2_{G_1}(u)d^2_{G_2}(x)+2d_{G_1}(u)d_{G_2}(x)\\
&+2d^2_{G_1}(u)d_{G_2}(x)+d_{G_1}(u)d^2_{G_2}(x)]\\
&[d^2_{G_1}(v)+d^2_{G_2}(x)+d^2_{G_1}(v)d^2_{G_2}(x)+2d_{G_1}(v)d_{G_2}(x)+2d^2_{G_1}(v)d_{G_2}(x)+d_{G_1}(v)d^2_{G_2}(x)]\\
=\;& \sum_{x\in V(G_2)}\sum_{uv\in G_1}[6d^4_{G_2}(x)d_{G_1}(u)d_{G_1}(v)(d_1(u)+d_{G_1}(v))\\
&+6d^3_{G_2}(x)d_{G_1}(u)d_{G_1}(v)(d_{G_1}(u)+d_{G_1}(v))+6d^2_{G_2}(x)d_{G_1}(u)d_{G_1}(v)(d_{G_1}(u)+d_{G_1}(v))\\
&+2d^2_{G_2}(x)d_{G_1}(u)d_{G_1}(v)(d_{G_1}(u)+d_{G_1}(v))+d^4_{G_2}(x)d^2_{G_1}(u)d^2_{G_1}(v)\\
&+4d^3_{G_2}(x)d^2_{G_1}(u)d^2_{G_1}(v)+6d^2_{G_2}(x)d^2_{G_1}(u)d^2_{G_1}(v)+d^4_{G_2}(x)\\
&+4d_{G_2}(x)d^2_{G_1}(u)d^2_{G_1}(v)+d^4_{G_2}(x)(d^2_{G_1}(u)+d^2_{G_1}(v))+2d^4_{G_2}(x)(d^2_{G_1}(u)+d^2_{G_1}(v))\\
&+d^2_{G_2}(x)(d^2_{G_1}(u)+d^2_{G_1}(v))+4d^4_{G_2}(x)d_{G_1}(u)d_{G_1}(v)+8d^3_{G_2}(x)d_{G_1}(u)d_{G_1}(x)\\
&+4d^2_{G_2}(x)d_{G_1}(u)d_{G_1}(v)+2d^4_{G_2}(x)(d_{G_1}(u)+d_{G_1}(v))+2d^3_{G_2}(x)(d_{G_1}(u)+d_{G_1}(v))]\\
=\;& 4F_2(G_2)GO_2(G_1)+12F(G_2)GO_2(G_1)+12M_1(G_2)GO_2(G_1)+8m_2GO_2(G_1)\\
&+12M_1(G_2)HM_2(G_1)+16m_2HM_2(G_1)+2F_2(G_2)HM_2(G_1)+8F(G_2)HM_2(G_1)\\
&+2n_2HM_2(G_1)+2F_2(G_2)F(G_1)+4F(G_2)F(G_1)+2M_1(G_2)F(G_1)\\
&+8F_2(G_2)M_2(G_1)+16F(G_2)M_2(G_1)+8M_1(G_2)M_2(G_1)+4F_2(G_2)M_1(G_1)\\
&+4F(G_2)M_1(G_1)+2m_1F_2(G_2)\\
=\;& 4GO_2(G_1)[2m_2+3M_1(G_2)+3F(G_2)+3F_2(G_2)]\\
&+2HM_2(G_1)[n_2+8m_2+6M_1(G_2)+4F(G_2)+F_2(G_2)]\\
&+2F(G_1)[M_1(G_2)+2F(G_2)+F_2(G_2)]+8M_2(G_1)[M_1(G_2)+2F(G_2)+F_2(G_2)]\\
&+4M_1(G_1)[F(G_2)+F_2(G_2)+2m_1F_2(G_2)]\\
S_3 =\;& \sum_{xy\in G_2}\sum_{uv\in G_1}[d_{G_1}(u)d_{G_2}(x)+d_{G_1}(u)d_{G_2}(x)]^2[d_{G_1}(v)+d_{G_2}(y)+d_{G_1}(v)d_{G_2}(y)]^2
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{xy\in G_2}\sum_{uv\in G_1}[d^2_{G_1}(u)+d^2_{G_2}(x)+d^2_{G_1}(u)d^2_{G_2}(x)+2d_{G_1}(u)d_{G_2}(x)+2d^2_{G_1}(u)d_{G_2}(x)\\
&\quad +2d_{G_1}(u)d^2_{G_2}(x)]\\
&\quad [d^2_{G_1}(v)+d^2_{G_2}(y)+d^2_{G_1}(v)d^2_{G_2}(y)+2d_{G_1}(v)d_{G_2}(y)+2d^2_{G_1}(v)d_{G_2}(y)+2d_{G_1}(v)d^2_{G_2}(y)]\\
&= \sum_{xy\in G_2}\sum_{uv\in G_1}[2d^2_{G_1}(u)d^2_{G_1}(v)d_{G_2}(x)d_{G_2}(y)(d_{G_2}(x)+d_{G_2}(y))\\
&\quad +2d^2_{G_2}(x)d^2_{G_2}(y)d_{G_1}(u)d_{G_1}(v)(d_{G_1}(u)+d_{G_1}(v))\\
&\quad +4(d_{G_1}(u)d^2_{G_1}(v)d^2_{G_2}(x)d_{G_2}(y)+d^2_{G_1}(u)d_{G_1}(v)d_{G_2}(x)d^2_{G_2}(y))\\
&\quad +2d^2_{G_1}(u)d_{G_1}(v)d^2_{G_2}(x)d_{G_2}(y)+d_{G_1}(u)d^2_{G_1}(v)d_{G_2}(x)d^2_{G_2}(y)\\
&\quad +4d_{G_1}(u)d_{G_1}(v)d_{G_2}(x)d_{G_2}(y)(d_{G_2}(x)+d_{G_2}(y))\\
&\quad +4d_{G_2}(x)d_{G_2}(y)d_{G_1}(u)d_{G_1}(v)(d_{G_1}(u)+d_{G_1}(v))\\
&\quad +2(d^2_{G_1}(u)d_{G_2}(x)d^2_{G_2}(y)+d^2_{G_1}(v)d^2_{G_2}(x)d_{G_2}(y))\\
&\quad +2(d_{G_1}(u)d_{G_2}(x)d^2_{G_2}(y)+d_{G_1}(v)d^2_{G_2}(x)d_{G_2}(y))\\
&\quad +2(d^2_{G_1}(u)d_{G_1}(v)d_{G_2}(y)+d_{G_1}(u)d^2_{G_1}(v)d_{G_2}(x))\\
&\quad +2(d^2_{G_1}(u)d_{G_1}(v)d^2_{G_2}(y)+d_{G_1}(u)d^2_{G_1}(v)d^2_{G_2}(x))\\
&\quad +d^2_{G_1}(u)d^2_{G_1}(v)d^2_{G_2}(x)d^2_{G_2}(y)+4d^2_{G_1}(u)d^2_{G_1}(v)d_{G_2}(x)d_{G_2}(y)\\
&\quad +4d_{G_1}(u)d_{G_1}(v)d^2_{G_2}(x)d^2_{G_2}(y)+d^2_{G_1}(u)d^2_{G_1}(v)(d^2_{G_2}(x)+d^2_{G_2}(y))\\
&\quad +d^2_{G_2}(x)d^2_{G_2}(y)(d^2_{G_1}(u)+d^2_{G_1}(v))+2d^2_{G_1}(u)d^2_{G_1}(v)(d_{G_2}(x)+d_{G_2}(y))\\
&\quad +2d^2_{G_2}(x)d^2_{G_2}(y)(d_{G_1}(u)+d_{G_1}(v))+d^2_{G_1}(u)d^2_{G_1}(v)+d^2_{G_2}(x)d^2_{G_2}(y)\\
&\quad +d^2_{G_1}(u)d^2_{G_2}(y)+d^2_{G_1}(v)d^2_{G_2}(x)+4d_{G_1}(u)d_{G_1}(v)d_{G_2}(x)d_{G_2}(y)]\\
&= 8HM_2(G_1)GO_2(G_2)+8HM_2(G_2)GO_2(G_1)+8GO_2(G_1)GO_2(G_2)\\
&\quad +4GO_2(G_1)GO_2(G_2)+16M_2(G_1)GO_2(G_2)+16M_2(G_2)GO_2(G_1)\\
&\quad +4F(G_1)GO_2(G_2)+4M_1(G_1)GO_2(G_2)+4M_1(G_2)GO_2(G_1)+4F(G_2)GO_2(G_1)\\
&\quad +4HM_2(G_1)HM_2(G_2)+16HM_2(G_1)M_2(G_2)+16M_2(G_1)HM_2(G_2)\\
&\quad +4HM_2(G_1)F(G_2)+4HM_2(G_2)F(G_1)+8HM_2(G_1)M_1(G_2)\\
&\quad +8HM_2(G_2)M_1(G_1)+2m_2HM_2(G_1)+2m_1HM_2(G_2)+2F(G_1)F(G_2)\\
&\quad +16M_2(G_1)M_2(G_2)\\
&= 12GO_2(G_1)GO_2(G_2)+4HM_2(G_1)HM_2(G_2)+2F(G_1)F(G_2)\\
&\quad +16M_2(G_1)M_2(G_2)+4GO_2(G_2)[M_1(G_1)+4M_2(G_1)+F(G_1)+2HM_2(G_1)]\\
&\quad +4GO_2(G_1)[M_1(G_2)+4M_2(G_2)+F(G_2)+2HM_2(G_2)]\\
&\quad +4HM_2(G_2)[m_1+4M_1(G_1)+8M_2(G_1)+2F(G_1)]\\
&\quad +4HM_2(G_1)[m_2+4M_1(G_2)+8M_2(G_2)+2F(G_2)]
\end{aligned}
$$

Adding the above inequalities we get the desired result. ∎

2.5 The Second Gourava Index of Strong Product of Graphs

In this section, we find the exact value of the second Gourava index of strong product of graphs.

Theorem 2.5. *Let $G_i, i = 1, 2$ be a (n_i, m_i) graph. Then*

$$\begin{aligned}
GO_2(G_1 \boxtimes G_2) &= 2m_2F(G_1) + 2m_1F(G_2) + GO_2(G_2)[n_1 + 8m_1 + 2F(G_1) + 6M_1(G_1) \\
&\quad +4M_2(G_1)] + GO_2(G_1)[n_2 + 8m_2 + 2F(G_2) + 6M_1(G_2) + 4M_2(G_2)] \\
&\quad +HM(G_2)[2m_1 + 2M_1(G_1) + F(G_1)] \\
&\quad +HM(G_1)[2m_2 + 2M_1(G_2) + F(G_2)] \\
&\quad +2M_2(G_2)[2m_1 + 6M_1(G_1) + 3F(G_1) + 2M_2(G_1)] \\
&\quad +2M_2(G_1)[2m_2 + 2M_1(G_2) + 3F(G_2) + 2M_2(G_2)] + M_1(G_2)[3M_1(G_1) \\
&\quad +4F(G_1) + 4M_2(G_1)] + M_1(G_1)[3M_1(G_2) + 4F(G_2)]
\end{aligned}$$

Proof:

$$\begin{aligned}
2GO_2(G_1 \boxtimes G_2) &= \sum_{(u,x)(v,y)\in G_1\boxtimes G_2} d_{G_1\boxtimes G_2}(u,x)d_{G_1\boxtimes G_2}(v,y)[d_{G_1\boxtimes G_2}(u,x) + d_{G_1\boxtimes G_2}(v,y)] \\
&= \sum_{u\in V(G_1)} \sum_{xy\in G_2} [d_{G_1}(u) + d_{G_2}(x) + d_{G_1}(u)d_{G_2}(x)] \\
&\quad [d_{G_1}(u) + d_{G_2}(y) + d_{G_1}(u)d_{G_2}(y)] \\
&\quad [d_{G_1}(u) + d_{G_2}(x) + d_{G_1}(u)d_{G_2}(x) + d_{G_1}(u) + d_{G_2}(y) + d_{G_1}(u)d_{G_2}(y)] \\
&\quad + \sum_{x\in V(G_2)} \sum_{uv\in G_1} [d_{G_1}(u) + d_{G_2}(x) + d_{G_1}(u)d_{G_2}(x)] \\
&\quad [d_{G_1}(v) + d_{G_2}(x) + d_{G_1}(v)d_{G_2}(x)] \\
&\quad [d_{G_1}(u) + d_{G_2}(x) + d_{G_1}(u)d_{G_2}(x) + d_{G_1}(v) + d_{G_2}(x) + d_{G_1}(v)d_{G_2}(x)] \\
&\quad + \sum_{xy\in G_2} \sum_{uv\in G_1} [d_{G_1}(u) + d_{G_2}(x) + d_{G_1}(u)d_{G_2}(x)] \\
&\quad [d_{G_1}(v) + d_{G_2}(y) + d_{G_1}(v)d_{G_2}(y)] \\
&\quad [d_{G_1}(u) + d_{G_2}(x) + d_{G_1}(u)d_{G_2}(x) + d_{G_1}(v) + d_{G_2}(y) + d_{G_1}(v)d_{G_2}(y)] \\
&= S_1 + S_2 + S_3
\end{aligned}$$

where S_1, S_2 and S_3 are the sums of the above order.

$$\begin{aligned}
S_1 &= \sum_{u\in V(G_1)} \sum_{xy\in G_2} [d_{G_1}(u) + d_{G_2}(x) + d_{G_1}(u)d_{G_2}(x)][d_{G_1}(u) + d_{G_2}(y) + d_{G_1}(u)d_{G_2}(y)] \\
&\quad [2d_{G_1}(u) + d_{G_1}(u)(d_{G_2}(x) + d_{G_2}(y)) + d_{G_2}(x) + d_{G_2}(y)] \\
&= \sum_{u\in V(G_1)} \sum_{xy\in G_2} [d^2_{G_1}(u) + d_{G_1}(u)(d_{G_2}(x) + d_{G_2}(y)) + d^2_{G_1}(u)(d_{G_2}(x) + d_{G_2}(y)) \\
&\quad +2d_{G_1}(u)(d_{G_2}(x)d_{G_2}(y)) + d^2_{G_1}(u)d_{G_2}(x)d_{G_2}(y) + d_{G_2}(x)d_{G_2}(y)]
\end{aligned}$$

$$
\begin{aligned}
& [2d_{G_1}(u) + d_{G_2}(x) + d_{G_2}(y) + d_{G_1}(u)(d_{G_2}(x) + d_{G_2}(y))] \\
= & \sum_{u \in V(G_1)} \sum_{xy \in G_2} [2d^3_{G_1}(u) + 3d^2_{G_1}(u)(d_{G_2}(x) + d_{G_2}(y)) + 3d^3_{G_1}(u)(d_{G_2}(x) + d_{G_2}(y)) \\
& + d_{G_1}(u)(d_{G_2}(x) + d_{G_2}(y))^2 + 2d^2_{G_1}(u)(d_{G_2}(x) + d_{G_2}(y))^2 + d^3_{G_1}(u)(d_{G_2}(x) + d_{G_2}(y))^2 \\
& + 4d^2_{G_1}(u)d_{G_2}(x)d_{G_2}(y) + 2d_{G_1}(u)d_{G_2}(x)d_{G_2}(y)(d_{G_2}(x) + d_{G_2}(y)) \\
& + 2d^2_{G_1}(u)d_{G_2}(x)d_{G_2}(y)(d_{_2}(x) + d_{G_2}(y)) \\
& + 2d^3_{G_1}(u)d_{G_2}(x)d_{G_2}(y) + d^2_{G_1}(u)d_{G_2}(x)d_{G_2}(y)(d_{G_2}(x) + d_{G_2}(y)) \\
& + d^3_{G_1}(u)d_{G_2}(x)d_{G_2}(y)(d_{G_2}(x) + d_{G_2}(y)) + 2d_{G_1}(u)d_{G_2}(x)d_{G_2}(y) \\
& + d_{G_2}(x)d_{G_2}(y)(d_{G_2}(x) + d_{G_2}(y)) + d_{G_1}(u)d_{G_2}(x)d_{G_2}(y)(d_{G_2}(x) + d_{G_2}(y))] \\
= & 4m_2F(G_1) + 6M_1(G_1)M_1(G_2) + 6F(G_1)M_1(G_2) + 4MHM(G_2) + 4M_1(G_1)HM(G_2) \\
& + 2F(G_1)HM(G_2) + 8M_1(G_1)M_2(G_2) + 8m_1GO_2(G_2) + 6M_1(G_1)GO_2(G_2) \\
& + 4F(G_1)M_2(G_2) + 2F(G_1)GO_2 + 8m_1M_2(G_2) + 2n_1GO_2(G_2) + 4m_1GO_2(G_2) \\
S_2 = & \sum_{x \in V(G_2)} \sum_{uv \in G_1} [d_{G_1}(u) + d_{G_2}(x) + d_{G_1}(u)d_{G_2}(x)][d_{G_1}(v) + d_{G_2}(x) + d_{G_1}(v)d_{G_2}(x)] \\
& [d_{G_1}(u) + d_{G_1}(v) + 2d_{G_2}(x) + d_{G_2}(x)(d_{G_1}(u) + d_{G_1}(v))] \\
= & \sum_{x \in V(G_2)} \sum_{uv \in G_1} [d^2_{G_2}(x) + d_{G_2}(x)(d_{G_1}(u) + d_{G_1}(v)) + d^2_{G_2}(x)(d_{G_1}(u) + d_{G_1}(v)) \\
& + 2d_{G_2}(x)(d_{G_1}(u)d_{G_1}(v)) + d^2_{G_2}(x)d_{G_1}(u)d_{G_2}(v) + d_{G_1}(u)d_{G_1}(v)] \\
& [d_{G_1}(u) + d_{G_1}(v) + 2d_{G_2}(x) + d_{G_2}(x)(d_{G_1}(u) + d_{G_1}(v))] \\
= & \sum_{x \in V(G_2)} \sum_{uv \in G_1} [2d^3_{G_2}(x) + d^2_{G_2}(x)(d_{G_1}(u) + d_{G_1}(v)) + d^3_{G_2}(x)(d_{G_1}(u) + d_{G_1}(v)) \\
& + 2d^2_{G_2}(x)(d_{G_1}(u) + d_{G_1}(v)) + d_{G_2}(x)(d_{G_1}(u) + d_{G_1}(v))^2 d^2_{G_2}(x)(d_{G_1}(u) + d_{G_1}(v))^2 \\
& + 2d^3_{G_2}(x)(d_{G_1}(u) + d_{G_1}(v)) + d^2_{G_2}(x)(d_{G_1}(u) + d_{G_1}(v))^2 + d^3_{G_2}(x)(d_{G_1}(u) + d_{G_1}(v))^2 \\
& + 4d^2_{G_2}(x)d_{G_1}(u)d_{G_1}(v) + 2d_{G_2}(x)d_{G_1}(u)d_{G_1}(v)(d_{G_1}(u) + d_{G_1}(v)) \\
& + 2d^2_{G_2}(x)d_{G_1}(u)d_{G_1}(v)(d_{G_1}(u) + d_{G_1}(v)) + 2d^3_{G_2}(x)d_{G_1}(u)d_{G_1}(v) \\
& + d^2_{G_2}(x)d_{G_1}(u)d_{G_1}(v)(d_{G_1}(u) + d_{G_1}(v)) + d^3_{G_2}(x)d_{G_1}(u)d_{G_1}(v)(d_{G_1}(u) + d_{G_1}(v)) \\
& + 2d_{G_2}(x)d_{G_1}(u)d_{G_1}(v) + d_{G_1}(u)d_{G_1}(v)(d_{G_1}(u) + d_{G_1}(v)) \\
& + d_{G_2}(x)d_{G_1}(u)d_{G_1}(v)(d_{G_1}(u) + d_{G_1}(v))] \\
= & 4m_1F(G_2) + 6M_1(G_2)M_1(G_1) + 6F(G_2)M_1(G_1) + 4m_2HM(G_1) + 4M_1(G_2)HM(G_1) \\
& + 2F(G_2)HM(G_1) + 8M_1(G_2)M_2(G_1) + 8m_2GO_2(G_1) + 6M_1(G_2)GO_2(G_1) \\
& + 4F(G_2)M_2(G_1) + 2F(G_2)GO_2(G_1) + 8m_2M_2(G_1) + 2n_2GO_2(G_1) + 4m_2GO_2(G_1) \\
S_3 = & \sum_{xy \in G_2} \sum_{uv \in G_1} [d_{G_1}(u)d_{G_2}(x) + d_{G_1}(u)d_{G_2}(x)][d_{G_1}(v) + d_{G_2}(y) + d_{G_1}(v)d_{G_2}(y)] \\
& [d_{G_1}(u) + d_{G_2}(x) + d_{G_1}(u)d_{G_2}(x) + d_{G_1}(v) + d_{G_2}(y) + d_{G_1}(v)d_{G_2}(y)]
\end{aligned}
$$

$$
\begin{aligned}
= & \sum_{xy\in G_2}\sum_{uv\in G_1}[d_{G_1}(u)d_{G_1}(v)+d_{G_1}(u)d_{G_2}(y)+d_{G_2}(x)d_{G_2}(y)+d_{G_1}(v)d_{G_2}(x)\\
& +d_{G_1}(u)d_{G_1}(v)(d_{G_2}(x)+d_{G_2}(y))+d_{G_2}(x)d_{G_2}(y)(d_{G_1}(u)+d_{G_1}(v))\\
& +d_{G_1}(u)d_{G_1}(v)d_{G_2}(x)d_{G_2}(y)][d_{G_1}(u)+d_{G_2}(x)+d_{G_1}(u)d_{G_2}(x)+d_{G_1}(v)\\
& +d_{G_2}(y)+d_{G_1}(v)d_{G_2}(y)]\\
= & \sum_{xy\in G_2}\sum_{uv\in G_1}[d^2_{G_1}(u)d_{G_1}(v)+d_{G_1}(u)d_{G_1}(v)d_{G_2}(x)+d^2_{G_1}(u)d_{G_1}(v)d_{G_2}(x)+d_{G_1}(u)d^2_{G_1}(v)\\
& +d_{G_1}(u)d_{G_1}(v)d_{G_2}(y)+d_{G_1}(u)d^2_{G_1}(v)d_{G_2}(y)+d^2_{G_1}(u)d_{G_2}(x)d_{G_2}(y)\\
& +d_{G_1}(u)d_{G_1}(v)d_{G_2}(y)+d_{G_1}(u)d^2_{G_2}(y)+d_{G_1}(u)d_{G_1}(v)d^2_{G_2}(y)+d_{G_1}(u)d_{G_1}(v)d_{G_2}(x)\\
& +d_{G_1}(v)d^2_{G_2}(x)+d_{G_1}(u)d_{G_1}(v)d^2_{G_2}(x)+d^2_{G_1}(v)d_{G_2}(x)+d_{G_1}(v)d_{G_2}(x)d_{G_2}(y)\\
& +d^2_{G_1}(v)d_{G_2}(x)d_{G_2}(y)+d_{G_1}(u)d_{G_2}(x)d_{G_2}(y)+d^2_{G_2}(x)d_{G_2}(y)+d_{G_1}(u)d^2_{G_2}(x)d_{G_2}(y)\\
& +d_{G_1}(v)d_{G_2}(x)d_{G_2}(y)+d_{G_2}(x)d^2_{G_2}(y)+d_{G_1}(v)d_{G_2}(x)d^2_{G_2}(y)\\
& +d^2_{G_1}(u)d_{G_1}(v)(d_{G_2}(x)+d_{G_2}(y))+d_{G_1}(u)d_{G_1}(v)(d^2_{G_2}(x)+d_{G_2}(x)d_{G_2}(y))\\
& +d^2_{G_1}(u)d_{G_1}(v)(d^2_{G_2}(x)+d_{G_2}(x)d_{G_2}(y))+d_{G_1}(u)d^2_{G_1}(v)(d_{G_2}(x)+d_{G_2}(y))\\
& +d_{G_1}(u)d_{G_1}(v)(d_{G_2}(x)d_{G_2}(y)+d^2_{G_2}(y))+d_{G_1}(u)d^2_{G_1}(v)(d_{G_2}(x)d_{G_2}(y)+d^2_{G_2}(y))\\
& +d_{G_2}(x)d_{G_2}(y)(d^2_{G_1}(u)+d_{G_1}(u)d_{G_1}(v))+d^2_{G_2}(x)d_{G_2}(y)(d_{G_1}(u)+d_{G_1}(v))\\
& +d^2_{G_2}(x)d_{G_2}(y)(d^2_{G_1}(u)+d_{G_1}(u)d_{G_1}(v))\\
& +d_{G_2}(x)d_{G_2}(y)(d_{G_1}(u)d_{G_1}(v)+d^2_{G_1}(v))+d^2_{G_1}(u)d_{G_1}(v)d_{G_2}(x)d_{G_2}(y)\\
& +d_{G_1}(u)d_{G_1}(v)d^2_{G_2}(x)d_{G_2}(y)+d^2_{G_1}(u)d_{G_1}(v)d^2_{G_2}(x)d_{G_2}(y)\\
& +d_{G_1}(u)d^2_{G_1}(v)d_{G_2}(x)d_{G_2}(y)+d_{G_1}(u)d_{G_1}(v)d_{G_2}(x)d^2_{G_2}(y)\\
& +d_{G_1}(u)d^2_{G_1}(v)d_{G_2}(x)d^2_{G_2}(y)]\\
= & \ 4m_2GO_2(G_1)+4m_1GO_2(G_2)+8M_2(G_1)M_1(G_2)+8M_1(G_1)M_2(G_2)\\
& +2F(G_1)M_1(G_2)+2F(G_2)M_1(G_1)+4F(G_1)M_2(G_2)+4F(G_2)M_2(G_1)\\
& +6GO_2(G_1)M_1(G_2)+6M_1(G_1)GO_2(G_2)+4M_2(G_1)(F(G_2)+2M_2(G_2))\\
& +2GO_2(G_1)(F(G_2)+2M_2(G_2))+4M_2(G_2)(F(G_1)+2M_2(G_1))\\
& +2GO_2(G_2)(F(G_1)+2M_2(G_1))+4GO_2(G_1)M_2(G_2)+4M_2(G_1)GO_2(G_2)\\
& +2GO_2(G_1)GO_2(G_2)
\end{aligned}
$$

Add this inequalities we ge the desired result. ∎

References

[1] M. R. Farahani, M. R. Rajesh Kanna and R. Pradeep Kumar, On the hyper-Zagreb indices of some nanostructures, *Asian Academic Research Journal of Multidisciplinary*, Vol. 3(1) (2016), 115–123.

[2] B. Furtula and I. Gutman, A forgotten topological index, *Journal of Mathematical Chemistry*, Vol. 53(4) (2015), 1184–1190.

[3] I. Gutman and N. Trinajstic, Graph theory and molecular orbitals. Total π-electron energy of alternant hydrocarbons, *Chemical Physics Letters*, Vol. 17(4) (1972), 535–538.

[4] R. Hammack, W. Imrich and S. Klavzar, Handbook of Product Graphs, CRC Press, (2011).

[5] V. R. Kulli, The Gourava indices and coindices of graphs, *Annals of Pure and Applied Mathematics*, Vol. 14(1) (2017), 33–38.

[6] R. Muruganandam and R. S. Manikandan, Gutman index of some graph operations, *International Journal of Applied Graph Theory*, Vol. 2 (2017), 1–20.

[7] G. H. Shirdel, H. Rezapour and A. M. Sayadi, The hyper-Zagreb index of graph operations, *Iranian Journal of Mathematical Chemistry*, Vol. 4(2) (2013), 213–220.

[8] H. Wiener, Structural determination of the paraffin boiling points, *Journal of the American Chemical Society*, Vol. 69 (1947), 17–20.