International Journal of Applied Graph Theory
Vol. 10, No. 2 (2026), 15 - 32. ISSN(Online) : 2456 – 7884

# Distance Based Reformulated First Zagreb Index of Some Graph Products

**M. Aruvi**
Department of Mathematics
University College of Engineering - BIT Campus
Anna University, Tiruchirappalli
Tamilnadu, India.
aruvim.aut@gmail.com

**Abstract**

In this work, a distance-based reformulation of the first Zagreb index is introduced. The exact values of this index are derived for several graph products, including the Cartesian, strong, corona, and tensor products of two graphs.

**Key words:** Topological indices, Graph operations

**2020 Mathematics Subject Classification** : 05C07, 05C12, 05C76

## 1 Introduction

We express the degree of a vertex $v$ in $G$ is $d_G(v)$. The distance between the shortest $(u, v)$-path in $G$ is symbolized by $d_G(u, v)$, for any $u, v \in V(G)$.

For basic definitions and notations see [2].

A topological index is a real number that is connected to a molecule's structural graph; it is unrelated to how the structure graph is represented. H.Wiener [14] in 1947, introduced a topological index based on the distance $d_G(u, v)$ which is named as Wiener index and it is established as

$$W(G) = \sum_{\{u,v\}\subseteq V(G)} d_G(u, v) = \frac{1}{2} \sum_{u,v\in V(G)} d_G(u, v).$$

Numerous research have been conducted to determine correlations between a molecule's structure and its properties using the Wiener index, which is based on the distance between any two vertices of $G$. Additionally, chemical compound biological activity is predicted using this method, see for reference [13].

There are some degree based topological indices of a graph which are known as Zagreb indices, established by Gutman et al. in [6]. The first Zagreb index $M_1(G)$ and the second Zagreb index $M_2(G)$ of a graph $G$ are established respectively:

$$M_1(G) = \sum_{uv\in E(G)} [d_G(u) + d_G(v)] = \sum_{v\in V(G)} d_G^2(v)$$

Ψ Received on July 14, 2026 / Revised on September 08, 2026 / Accepted on September 09, 2026

$$M_2(G) = \sum_{uv \in E(G)} d_G(u) d_G(v)$$

Dobrynin et.al. in [3] and Gutman in [5] considered the molecular Topological index of a graph as a weighted basedthe Wiener index and compared its properties with the properties of Wiener index.

The degree distance $DD(G)$ is established by

$$DD(G) = \sum_{\{u,v\} \subseteq V(G)} d_G(u,v)[d_G(u) + d_G(v)] = \frac{1}{2} \sum_{u,v \in V(G)} d_G(u,v)[d_G(u) + d_G(v)].$$

The Schultz index of the second kind which is known as the Gutman index is defined by Gutman [5] and it is denoted as Gut(G). The Gutman index is established as

$$Gut(G) = \sum_{\{u,v\} \subseteq V(G)} d_G(u,v)[d_G(u) d_G(v)] = \frac{1}{2} \sum_{u,v \in V(G)} d_G(u,v)[d_G(u) d_G(v)].$$

In [6], the sum of cubes of vertex degrees was involved in the investigation of the total $\pi-$ electron energy and it was again investigated by B.Furtula et. al. in [4] as Forgotten Topological index or F-index . It is established for a graph $G$ as

$$F(G) = \sum_{v \in V(G)} d_G(v)^3 = \sum_{uv \in E(G)} [d_G(u)^2 + d_G(v)^2].$$

Nilanjan De et. al. in [10] derived the exact values for $F-$ index of certain graph operations. Muruganandam et al. in [9] computed the exact values of the distance version of F- index, and established as

$$\begin{aligned} DF(G) &= \sum_{\{u,v\} \subseteq V(G)} d_G(u,v)[d_G(u)^2 + d_G(v)^2] \\ &= \frac{1}{2} \sum_{u,v \in V(G)} d_G(u,v)[d_G(u)^2 + d_G(v)^2], \end{aligned}$$

where the sum is taken over all pairs $(u, v)$ of vertices of $G$.

Shirdel et.al. in [12] found Hyper-Zagreb index $HM(G)$ which is established as

$$HM(G) = \sum_{uv \in E(G)} [d_G(u) + d_G(v)]^2.$$

Also they have computed the hyper - Zagreb index of the cartesian product, composition, join and disjuction of graphs.

M.Aruvi et.al. in [1] have established the distance version of Hyper-Zagreb index of a graph and symbolized by $DHM(G)$, so that

$$DHM(G) = \sum_{\{u,v\} \subseteq V(G)} d_G(u,v)[d_G(u) + d_G(v)]^2 = \frac{1}{2} \sum_{u,v \in V(G)} d_G(u,v)[d_G(u) + d_G(v)]^2.$$

Milicevic et.al [8] in 2004 reformulated the Zagreb indices interms of edge - degrees instead of vertex - degrees:

$$EM_1(G) = \sum_{e \in E} d_G(e)^2$$
$$EM_2(G) = \sum_{e \sim f} d_G(e) d_G(f)$$

Nilanjan De. et. al.[11] investigated the behaviour of the reformulated first Zagreb index, and they also applied their findings to several chemically intriguing molecular graphs and nano-structures.

In this paper, the distance-based reformulated first Zagreb index is introduced as

$$\begin{aligned} DEM_1(G) &= \sum_{\{u,v\} \subseteq V(G)} d_G(u,v)[d_G(u) + d_G(v) - 2]^2 \\ &= \frac{1}{2} \sum_{u,v \in V(G)} d_G(u,v)[d_G(u) + d_G(v) - 2]^2. \end{aligned}$$

Additionally, the precise values of the distance-based reformulated first Zagreb index for the Cartesian, strong, corona, and tensor products of two graphs are obtained.

## 2 Main Results

### 2.1 Distance Based Reformulated First Zagreb Index of Cartesian Product of Graphs

In this section, we compute the distance based reformulated first Zagreb index of cartesian product $G_1 \Box G_2$ of the graphs $G_1$ and $G_2$.

**Lemma 2.1.** [7] *Let $w_{il} = (u_i, v_l)$ and $w_{mn} = (u_m, v_n)$ be in $V(G_1 \Box G_2)$. Then the degree of the vertex $w_{il}$ is $d_{(G_1 \Box G_2)}(w_{il}) = d_{G_1}(u_i) + d_{G_2}(v_l)$*

**Lemma 2.2.** [7] *Let $w_{il} = (u_i, v_l)$ and $w_{mn} = (u_m, v_n)$ be in $V(G_1 \boxtimes G_2)$. Then the distance between $w_{il}$ and $w_{mn}$ is $d_{(G_1 \Box G_2)}(w_{il}, w_{mn}) = d_{G_1}(u_i, u_m) + d_{G_2}(v_l, v_n)$*

**Theorem 2.3.** *Let $G_i, i = 1, 2$, be a $(p_i, q_i)$- graph. Then*

$$\begin{aligned} 2 \times DEM_1(G) &= 2p_2 DEM_1(G_1) + 16q_2 DD(G_1) + 8W(G_1)[M_1(G_2) - 4q_2] \\ &+ 2p_1 DEM_1(G_2) + 16q_1 DD(G_2) + 8W(G_2)\Big[M_1(G_1) - 4q_1\Big] \\ &= 2p_2(n_2 - 1)DEM_1(G_1) + 4W(G_1)[(n_2 - 1)M_1(G_2) + 4q_2^2 + M_1(G_2)] \\ &+ 16q_2(p_2 - 1)DD(G_1) - 32q_2(p_2 - 1)W(G_1) \\ &+ 4W(G_2)[(p_1 - 1)M_1(G_1) + 4q_1^2 - M_1(G_1) + 2p_1(p_1 - 1) - 8q_1(n_1 - 1)] \\ &+ 2p_1(p_1 - 1)DH(G_2) + 16q_1(p_1 - 1)DD(G_2) - 8p_1(p_1 - 1)DD(G_2) \end{aligned}$$

**Proof:** Let $G = G_1 \Box G_2$. Then,

$$\begin{aligned}
2 \times DEM_1(G) &= \sum_{w_{il}, w_{mn} \in V(G)} d_G(w_{il}, w_{mn})[d_G(w_{il}) + d_G(w_{mq}) - 2]^2 \\
&= \sum_{l=0}^{p_2-1} \sum_{i,m=0, i\neq m}^{p_1-1} d_G(w_{il}, w_{ml})[d_G(w_{il}) + d_G(w_{ml}) - 2]^2 \\
&+ \sum_{i=0}^{p_1-1} \sum_{l,q=0, l\neq m}^{p_2-1} d_G(w_{il}, w_{iq})[d_G(w_{il}) + d_G(w_{iq}) - 2]^2 \\
&+ \sum_{l,q=0, l\neq q}^{p_2-1} \sum_{i,m=0, i\neq m}^{p_1-1} d_G(w_{il}, w_{mn})[d_G(w_{il}) + d_G(w_{mn}) - 2]^2 \\
&= A_1 + A_2 + A_3,
\end{aligned}$$

where $A_1,\ A_2,\ A_3$ are the sums of the above terms in order. Now we calculate $A_1,\ A_2,$ and $A_3$ separately.

First we calculate $A_1$.

$$\begin{aligned}
A_1 &= \sum_{l=0}^{p_2-1} \sum_{i,m=0, i\neq m}^{p_1-1} d_G(w_{il}, w_{ml})[d_G(w_{il}) + d_G(w_{ml}) - 2]^2 \\
&= \sum_{l=0}^{p_2-1} \sum_{i,m=0, i\neq m}^{p_1-1} d_{G_1}(u_i, u_m)\Big[d_{G_1}(u_i) + d_{G_2}(v_l) + d_{G_1}(u_m) + d_{G_2}(v_l) - 2\Big]^2 \\
&= \sum_{l=0}^{p_2-1} \sum_{i,m=0, i\neq m}^{p_1-1} d_{G_1}(u_i, u_m)\Big[d_{G_1}(u_i) + 2d_{G_2}(v_l) + d_{G_1}(u_m) - 2\Big]^2 \\
&= \sum_{l=0}^{p_2-1} \sum_{i,m=0, i\neq m}^{p_1-1} d_{G_1}(u_i, u_m)\Big[d^2_{G_1}(u_i) + d^2_{G_1}(u_m) + 4d^2_{G_2}(v_l) + 4 + 2d_{G_1}(u_i)d_{G_1}(u_m) \\
&\quad +4d_{G_1}(u_i)d_{G_2}(v_l) - 4d_{G_1}(u_i) + 4d_{G_2}(u_m)d_{G_2}(v_l) - 4d_{G_1}(u_m) - 8d_{G_2}(v_l)\Big] \\
&= \sum_{i,m=0, i\neq m}^{p_1-1} d_{G_1}(u_i, u_m)[d_{G_1}(u_i) + d_{G_1}(u_m) - 2]^2 \sum_{l=0}^{p_2-1} 1 \\
&\quad +4 \sum_{i,m=0, i\neq m}^{p_1-1} d_{G_1}(u_i, u_m)[d_{G_1}(u_i) + d_{G_1}(u_m)] \sum_{l=0}^{p_2-1} d_{G_2}(v_l) \\
&\quad -8 \sum_{i,m=0, i\neq m}^{p_1-1} d_{G_1}(u_i, u_m) \sum_{l=0}^{p_2-1} d_{G_2}(v_l) \\
&\quad +4 \sum_{i,m=0, i\neq m}^{p_1-1} d_{G_1}(u_i, u_m) \sum_{l=0}^{p_2-1} d^2_{G_2}(v_l) \\
&= 2p_2 DEM_1(G_1) + 16q_2 DD(G_1) + 8W(G_1)[M_1(G_2) - 4q_2]
\end{aligned}$$

Next we calculate $A_2$.

$$\begin{aligned}
A_2 &= \sum_{i=0}^{p_1-1} \sum_{l,n=0,l\neq n}^{p_2-1} d_G(w_{il},w_{in})[d_G(w_{il})+d_G(w_{in})-2]^2 \\
&= \sum_{i=0}^{p_1-1} \sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l,v_n)\Big[d_{G_1}(u_i)+d_{G_2}(v_l)+d_{G_1}(u_i)+d_{G_2}(v_n)-2\Big]^2 \\
&= \sum_{i=0}^{p_1-1} \sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l,v_n)\Big[2d_{G_1}(u_i)+d_{G_2}(v_l)+d_{G_2}(v_n)-2\Big]^2 \\
&= \sum_{i=0}^{p_1-1} \sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l,v_n)\Big[d^2_{G_2}(v_l)+d^2_{G_2}(v_n)+4d^2_{G_1}(u_i)+4+2d_{G_2}(v_l)d_{G_2}(v_n) \\
&\quad +4d_{G_1}(u_i)d_{G_2}(v_l)-4d_{G_2}(v_l)+4d_{G_1}(u_i)d_{G_2}(v_n)-4d_{G_2}(v_n)-8d_{G_1}(u_i)\Big] \\
&= \sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l,v_n)[d_{G_2}(v_l)+d_{G_2}(v_n)-2]^2 \sum_{i=0}^{p_1-1} 1 \\
&\quad +4\sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l,v_n)[d_{G_2}(v_l)+d_{G_2}(v_n)] \sum_{i=0}^{p_1-1} d_{G_1}(u_i) \\
&\quad -8\sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l,v_n) \sum_{i=0}^{p_1-1} d_{G_1}(u_i) + 4\sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l,v_n) \sum_{i=0}^{p_1-1} d^2_{G_1}(u_i) \\
&= 2p_1DEM_1(G_2)+16q_1DD(G_2)+8W(G_2)\Big[M_1(G_1)-4q_1\Big]
\end{aligned}$$

Finally we compute $A_3$.

$$\begin{aligned}
A_3 &= \sum_{l,n=0,l\neq n}^{p_2-1} \sum_{i,m=0,i\neq m}^{p_1-1} d_G(w_{il},w_{mn})[d_G(w_{il})+d_G(w_{mn})-2]^2 \\
&= \sum_{l,n=0,l\neq n}^{p_2-1} \sum_{i,m=0,i\neq m}^{p_1-1} \Big(d_{G_1}(u_i,u_m)+d_{G_2}(v_l,v_n)\Big) \\
&\quad \Big[d_{G_1}(u_i)+d_{G_2}(v_l)+d_{G_1}(u_m)+d_{G_2}(v_n)-2\Big]^2 \\
&= \sum_{l,n=0,l\neq n}^{p_2-1} \sum_{i,m=0,i\neq m}^{p_1-1} \Big(d_{G_1}(u_i,u_m)+d_{G_2}(v_l,v_n)\Big)\Big[d^2_{G_1}(u_i)+d^2_{G_2}(v_l)+d^2_{G_1}(u_m) \\
&\quad +d^2_{G_2}(v_n)+4+2d_{G_1}(u_i)d_{G_2}(v_l)+2d_{G_1}(u_i)d_{G_1}(u_m)+2d_{G_1}(u_i)d_{G_2}(v_n) \\
&\quad -4d_{G_1}(u_i)+d_{G_1}(u_m)d_{G_2}(v_l)+2d_{G_2}(v_l)d_{G_2}(v_n)-4d_{G_2}(v_l)+2d_{G_1}(u_m)d_{G_2}(v_n) \\
&\quad -4d_{G_1}(u_m)-4d_{G_2}(v_n)\Big] \\
&= \sum_{l,n=0,l\neq n}^{p_2-1} \sum_{i,m=0,i\neq m}^{p_1-1} \Big(d_{G_1}(u_i,u_m)+d_{G_2}(v_l,v_n)\Big)\Big[[d_{G_1}(u_i)+d_{G_1}(u_m)-2]^2 \\
&\quad +[d_{G_2}(v_l)+d_{G_2}(v_n)]^2+2[d_{G_1}(u_i)+d_{G_1}(u_m)][d_{G_2}(v_l)+d_{G_2}(v_n)] \\
&\quad -4[d_{G_2}(v_l)+d_{G_2}(v_n)]\Big]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{l,n=0,l\neq n}^{p_2-1} \sum_{i,m=0,i\neq m}^{p_1-1} \Big\{ d_{G_1}(u_i,u_m)\Big[[d_{G_1}(u_i)+d_{G_1}(u_m)-2]^2+[d_{G_2}(v_l)+d_{G_2}(v_n)]^2 \\
&\quad +2[d_{G_1}(u_i)+d_{G_1}(u_m)][d_{G_2}(v_l)+d_{G_2}(v_n)]-4[d_{G_2}(v_l)+d_{G_2}(v_n)]\Big] \\
&\quad +d_{G_2}(v_l,v_n)\Big[[d_{G_1}(u_i)+d_{G_1}(u_m)-2]^2+[d_{G_2}(v_l)+d_{G_2}(v_n)]^2 \\
&\quad +2[d_{G_1}(u_i)+d_{G_1}(u_m)][d_{G_2}(v_l)+d_{G_2}(v_n)]-4[d_{G_2}(v_l)+d_{G_2}(v_n)]\Big]\Big\} \\
&= \sum_{l,n=0,l\neq n}^{p_2-1} 1 \sum_{i,m=0,i\neq m}^{p_1-1} d_{G_1}(u_i,u_m)[d_{G_1}(u_i)+d_{G_1}(u_m)-2]^2 \\
&\quad +\sum_{l,n=0,l\neq n}^{p_2-1} [d_{G_2}(v_l)+d_{G_2}(v_n)]^2 \sum_{i,m=0,i\neq m}^{p_1-1} d_{G_1}(u_i,u_m) \\
&\quad +2\sum_{l,n=0,l\neq n}^{p_2-1} [d_{G_2}(v_l)+d_{G_2}(v_n)] \sum_{i,m=0,i\neq m}^{p_1-1} d_{G_1}(u_i,u_m)[d_{G_1}(u_i)+d_{G_1}(u_m)] \\
&\quad -4\sum_{l,n=0,l\neq n}^{p_2-1} [d_{G_2}(v_l)+d_{G_2}(v_n)] \sum_{i,m=0,i\neq m}^{p_1-1} d_{G_1}(u_i,u_m) \\
&\quad +\sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l,v_n) \sum_{i,m=0,i\neq m}^{p_1-1} [d_{G_1}(u_i)+d_{G_1}(u_m)-2]^2 \\
&\quad +\sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l,v_n)[d_{G_2}(v_l)+d_{G_2}(v_n)]^2 \sum_{i,m=0,i\neq m}^{p_1-1} 1 \\
&\quad +2\sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l,v_n)[d_{G_2}(v_l)+d_{G_2}(v_n)] \sum_{i,m=0,i\neq m}^{p_1-1} [d_{G_1}(u_i)+d_{G_1}(u_m)] \\
&\quad -4\sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l,v_n)[d_{G_2}(v_l)+d_{G_2}(v_n)] \sum_{i,m=0,i\neq m}^{p_1-1} 1 \\
&= 2p_2(n_2-1)DEM_1(G_1)+4W(G_1)[(n_2-1)M_1(G_2)+4q_2^2+M_1(G_2)] \\
&\quad +16q_2(p_2-1)DD(G_1)-32q_2(p_2-1)W(G_1) \\
&\quad +4W(G_2)[(p_1-1)M_1(G_1)+4q_1^2-M_1(G_1)+2p_1(p_1-1)-8q_1(p_1-1)] \\
&\quad +2p_1(p_1-1)DH(G_2)+16q_1(p_1-1)DD(G_2)-8p_1(p_1-1)DD(G_2)
\end{aligned}
$$

By adding $A_1$, $A_2$ and $A_3$, we get the desired result. ∎

## 2.2 Distance Based Reformulated First Zagreb Index of Strong Product of Graphs

In this section, we compute the distance based reformulated first Zagreb index of strong product $G_1 \boxtimes G_2$ of the graphs $G_1$ and $G_2$.

**Lemma 2.4.** [7] *The degree of the vertex* $(u_i, v_l)$ *of* $V(G_1 \boxtimes G_2)$ *is* $d_{G_1}(u_i) + d_{G_2}(v_l) + d_{G_1}(u_i)d_{G_2}(v_l)$, *That is* $d_{G_1\boxtimes G_2}(u_i, v_l) = d_{G_1}(u_i) + d_{G_2}(v_l) + d_{G_1}(u_i)d_{G_2}(v_l)$,

**Lemma 2.5.** *Let $w_{il} = (u_i, v_l)$ and $w_{mn} = (u_m, v_n)$ be in $V(G_1 \boxtimes G_2)$. Then the distance between $w_{il}$ and $w_{mn}$ is*

$$d_{G_1 \boxtimes G_2}(w_{il}, w_{mn}) = \begin{cases} d_{G_2}(v_l, v_n), & i = m, l \neq n \\ d_{G_1}(u_i, u_m), & i \neq m, l = n \\ d_{G_2}(v_l, v_n), & i \neq m, l \neq n \end{cases}$$

**Theorem 2.6.** *Let $G_i, i = 1, 2$, be a $(p_i, q_i)$- graph. Then*

$$\begin{aligned} 2 \times DE(G_1 \boxtimes G_2) &= 2p_2 DEM_1(G_1) + 8q_2 DHM(G_1) + 8M_1(G_2)DD(G_1) \\ &\quad +2M_1(G_2)DHM(G_1) - 32q_2 W(G_1) + 8M_1(G_2)W(G_1) \\ &\quad +2p_1 DEM_1(G_2) + 8q_1 DHM(G_2) + 8M_1(G_1)DD(G_2) \\ &\quad +2M_1(G_1)DHM(G_2) - 32q_1 W(G_2) + 8M_1(G_1)W(G_2) \\ &\quad +2p_1(p_1-1)DEM_1(G_2) + [4DD(G_2) + 4Gut(G_2)][4q_1^2 - M_1(G_1)] \\ &\quad +4W(G_2)[(p_1-1)M_1(G_1) + 4q_1^2 - M_1(G_2) - 8q_1(p_1-1)] \\ &\quad +8q_1(p_1-1)[2Gut(G_2) + DF(G_2)] \\ &\quad +2(p_1-1)M_1(G_1)[DF(G_2) + 2DD(G_2)] \end{aligned}$$

**Proof***: Let $G = G_1 \boxtimes G_2$. Then,*

$$\begin{aligned} 2 \times DEM_1(G) &= \sum_{w_{il}, w_{mn} \in V(G)} d_G(w_{il}, w_{mn}) \Big[ d_G(w_{il}) + d_G(w_{mn}) - 2 \Big]^2 \\ &= \sum_{l=0}^{p_2-1} \sum_{i,m=0, i \neq m}^{p_1-1} d_G(w_{il}, w_{ml}) \Big[ d_G(w_{il}) + d_G(w_{ml}) - 2 \Big]^2 \\ &+ \sum_{i=0}^{p_1-1} \sum_{l,n=0, l \neq m}^{p_2-1} d_G(w_{il}, w_{in}) \Big[ d_G^2(w_{il}) + d_G^2(w_{in}) - 2 \Big]^2 \\ &+ \sum_{l,n=0, l \neq n}^{p_2-1} \sum_{i,m=0, i \neq m}^{p_1-1} d_G(w_{il}, w_{mn}) \Big[ d_G^2(w_{il}) + d_G^2(w_{mn}) - 2 \Big]^2 \\ &= A_1 + A_2 + A_3, \end{aligned}$$

*where $A_1$, $A_2$, $A_3$ are the sums of the above terms in order. Now we calculate $A_1$, $A_2$, and $A_3$ separately. First we calculate $A_1$.*

$$\begin{aligned} A_1 &= \sum_{l=0}^{p_2-1} \sum_{i,m=0, i \neq m}^{p_1-1} d_G(w_{il}, w_{ml}) \Big[ d_G(w_{il}) + d_G(w_{ml}) - 2 \Big]^2 \\ &= \sum_{l=0}^{p_2-1} \sum_{i,m=0, i \neq m}^{p_1-1} d_{G_1}(u_i, u_m) \Big[ d_{G_1}(u_i) + d_{G_2}(v_l) + d_{G_1}(u_i) d_{G_2}(v_l) \\ &\quad + d_{G_1}(u_m) + d_{G_2}(v_l) + d_{G_1}(u_m) d_{G_2}(v_l) - 2 \Big]^2 \end{aligned}$$

$$
\begin{aligned}
&= \sum_{l=0}^{p_2-1} \sum_{i,m=0,i\neq m}^{p_1-1} d_{G_1}(u_i,u_m)\Big[d_{G_1}(u_i)+d_{G_1}(u_m)+2d_{G_2}(v_l)\\
&\quad +d_{G_2}(v_l)[d_{G_1}(u_i)+d_{G_1}(u_m)]-2\Big]^2\\
&= \sum_{l=0}^{p_2-1} \sum_{i,m=0,i\neq m}^{p_1-1} d_{G_1}(u_i,u_m)\Big[d^2_{G_1}(u_i)+d^2_{G_1}(u_m)+4d^2_{G_2}(v_l)\\
&\quad +d^2_{G_2}(v_l)[d_{G_1}(u_i)+d_{G_1}(u_m)]^2+2d_{G_1}(u_i)d_{G_1}(u_m)+4d_{G_1}(u_i)d_{G_2}(v_l)\\
&\quad +2d_{G_1}(u_i)d_{G_2}(v_l)[d_{G_1}(u_i)+d_{G_1}(u_m)]-4d_{G_1}(u_i)+4d_{G_1}(u_m)d_{G_2}(v_l)\\
&\quad +2d_{G_1}(u_m)d_{G_2}(v_l)[d_{G_1}(u_i)+d_{G_1}(u_m)]-4d_{G_1}(u_m)\\
&\quad +4d^2_{G_2}(v_l)[d_{G_1}(u_i)+d_{G_1}(u_m)]-8d_{G_2}(v_l)-4d_{G_2}(v_l)[d_{G_1}(u_i)+d_{G_1}(u_m)]\Big]\\
&= \sum_{l=0}^{p_2-1} \sum_{i,m=0,i\neq m}^{p_1-1} d_{G_1}(u_i,u_m)\Big[[d_{G_1}(u_i)+d_{G_1}(u_m)-2]^2\\
&\quad +2d_{G_2}(v_l)[d_{G_1}(u_i)+d_{G_1}(u_m)]^2+4d^2_{G_2}(v_l)[d_{G_1}(u_i)+d_{G_1}(u_m)]\\
&\quad +d^2_{G_2}(v_l)[d_{G_1}(u_i)+d_{G_1}(u_m)]^2-8d_{G_2}(v_l)+4d^2_{G_2}(v_l)\Big]\\
&= \sum_{l=0}^{p_2-1} 1 \sum_{i,m=0,i\neq m}^{p_1-1} d_{G_1}(u_i,u_m)[d_{G_1}(u_i)+d_{G_1}(u_m)-2]^2\\
&\quad +2\sum_{l=0}^{p_2-1} d_{G_2}(v_l) \sum_{i,m=0,i\neq m}^{p_1-1} d_{G_1}(u_i,u_m)[d_{G_1}(u_i)+d_{G_1}(u_m)]^2\\
&\quad +4\sum_{l=0}^{p_2-1} d^2_{G_2}(v_l) \sum_{i,m=0,i\neq m}^{p_1-1} d_{G_1}(u_i,u_m)[d_{G_1}(u_i)+d_{G_1}(u_m)]\\
&\quad +\sum_{l=0}^{p_2-1} d^2_{G_2}(v_l) \sum_{i,m=0,i\neq m}^{p_1-1} d_{G_1}(u_i,u_m)[d_{G_1}(u_i)+d_{G_1}(u_m)]^2\\
&\quad -8\sum_{l=0}^{p_2-1} d_{G_2}(v_l) \sum_{i,m=0,i\neq m}^{p_1-1} d_{G_1}(u_i,u_m)+4\sum_{l=0}^{p_2-1} d^2_{G_2}(v_l) \sum_{i,m=0,i\neq m}^{p_1-1} d_{G_1}(u_i,u_m)\\
&= 2p_2DEM_1(G_1)+8q_2DHM(G_1)+8M_1(G_2)DD(G_1)\\
&\quad +2M_1(G_2)DHM(G_1)-32q_2W(G_1)+8M_1(G_2)W(G_1)
\end{aligned}
$$

*Next we compute $A_2$.*

$$
\begin{aligned}
A_2 &= \sum_{i=0}^{p_1-1} \sum_{l,n=0,l\neq n}^{p_2-1} d_G(w_{il},w_{ml})\Big[d_G(w_{il})+d_G(w_{ml})-2\Big]^2\\
&= \sum_{i=0}^{p_1-1} \sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l,v_n)\Big[d_{G_1}(u_i)+d_{G_2}(v_l)+d_{G_1}(u_i)d_{G_2}(v_l)+d_{G_1}(u_i)\\
&\quad +d_{G_2}(v_n)+d_{G_1}(u_i)d_{G_2}(v_n)-2\Big]^2
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=0}^{p_1-1} \sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l,v_n)\Big[d_{G_2}(v_l)+d_{G_2}(v_n)+2d_{G_1}(u_i)\\
&\quad +d_{G_1}(u_i)[d_{G_2}(v_l)+d_{G_2}(v_n)]-2\Big]^2\\
&= \sum_{i=0}^{p_1-1} \sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l,v_n)\Big[d^2_{G_2}(v_l)+d^2_{G_2}(v_n)+4d^2_{G_1}(u_i)\\
&\quad +d^2_{G_1}(u_i)[d_{G_2}(v_l)+d_{G_2}(v_n)]^2+4+2d_{G_2}(v_l)d_{G_2}(v_n)+4d_{G_2}(v_l)d_{G_1}(u_i)\\
&\quad +2d_{G_2}(v_l)d_{G_1}(u_i)[d_{G_2}(v_l)+d_{G_2}(v_n)]-4d_{G_2}(v_l)+4d_{G_2}(v_n)d_{G_1}(u_i)\\
&\quad +2d_{G_2}(v_n)d_{G_1}(u_i)[d_{G_2}(v_l)+d_{G_2}(v_n)]-4d_{G_2}(v_n)\\
&\quad +4d^2_{G_1}(u_i)[d_{G_2}(v_l)+d_{G_2}(v_n)]-8d_{G_1}(u_i)-4d_{G_1}(u_i)[d_{G_2}(v_l)+d_{G_2}(v_n)]\Big]\\
&= \sum_{i=0}^{p_1-1} \sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l,v_n)\Big[[d_{G_2}(v_l)+d_{G_2}(v_n)-2]^2\\
&\quad +2d_{G_1}(u_i)[d_{G_2}(v_l)+d_{G_2}(v_n)]^2+4d^2_{G_1}(u_i)[d_{G_2}(v_l)+d_{G_2}(v_n)]\\
&\quad +d^2_{G_1}(u_i)[d_{G_2}(v_l)+d_{G_2}(v_n)]^2-8d_{G_2}(v_l)+4d^2_{G_2}(v_l)\Big]\\
&= \sum_{i=0}^{p_1-1} 1 \sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l,v_n)[d_{G_2}(v_l)+d_{G_2}(v_n)-2]^2\\
&\quad +2\sum_{i=0}^{p_1-1} d_{G_1}(u_i) \sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l,v_n)[d_{G_2}(v_l)+d_{G_2}(v_n)]^2\\
&\quad +4\sum_{i=0}^{p_1-1} d^2_{G_1}(u_i) \sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l,v_n)[d_{G_2}(v_l)+d_{G_2}(v_n)]\\
&\quad +\sum_{i=0}^{p_1-1} d^2_{G_1}(u_i) \sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l,v_n)[d_{G_2}(v_l)+d_{G_2}(v_n)]^2\\
&\quad -8\sum_{i=0}^{p_1-1} d_{G_1}(u_i) \sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l,v_n)+4\sum_{i=0}^{p_1-1} d^2_{G_1}(u_i) \sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l,v_n)\\
&= 2p_1DEM_1(G_2)+8q_1DHM(G_2)+8M_1(G_1)DD(G_2)\\
&\quad +2M_1(G_1)DHM(G_2)-32q_1W(G_2)+8M_1(G_1)W(G_2)
\end{aligned}
$$

*Finally we compute* $A_3$.

$$
\begin{aligned}
A_3 &= \sum_{l,n=0,l\neq n}^{p_2-1} \sum_{i,m=0,i\neq m}^{p_1-1} d_G(w_{il},w_{mn})[d_G(w_{il})^2+d_G(w_{mn})^2-2]^2\\
&= \sum_{l,n=0,l\neq n}^{p_2-1} \sum_{i,m=0,i\neq m}^{p_1-1} d_{G_2}(v_l,v_n)\Big[d_{G_1}(u_i)+d_{G_2}(v_l)+d_{G_1}(u_i)d_{G_2}(v_l)\\
&\quad +d_{G_1}(u_m)+d_{G_2}(v_n)+d_{G_1}(u_m)d_{G_2}(v_n)-2\Big]^2
\end{aligned}
$$

$$
\begin{aligned}
= & \sum_{l,n=0,l\neq n}^{p_2-1} \sum_{i,m=0,i\neq m}^{p_1-1} d_{G_2}(v_l,v_n)\Big[d^2_{G_1}(u_i)+d^2_{G_2}(v_l)+d^2_{G_1}(u_i)d^2_{G_2}(v_l) \\
& +d^2_{G_1}(u_m)+d^2_{G_2}(v_n)+d^2_{G_1}(u_m)d^2_{G_2}(v_n)+4+2d_{G_1}(u_i)d_{G_2}(v_l) \\
& +2d^2_{G_1}(u_i)d_{G_2}(v_l)+2d_{G_1}(u_i)d^2_{G_2}(v_l)+2d_{G_1}(u_m)d_{G_2}(v_n) \\
& +2d^2_{G_1}(u_m)d_{G_2}(v_n)+2d_{G_1}(u_m)d^2_{G_2}(v_n)+2d_{G_1}(u_i)d_{G_1}(u_m) \\
& +2d_{G_1}(u_i)d_{G_2}(v_n)+2d_{G_1}(u_i)d_{G_1}(u_m)d_{G_2}(v_n)+4d_{G_1}(u_i) \\
& +2d_{G_1}(u_m)d_{G_2}(v_l)+2d_{G_2}(v_l)d_{G_2}(v_n)+2d_{G_1}(u_m)d_{G_2}(v_l)d_{G_2}(v_n)-4d_{G_2}(v_l) \\
& +2d_{G_1}(u_i)d_{G_1}(u_m)d_{G_2}(v_l)+2d_{G_1}(u_i)d_{G_2}(v_l)d_{G_2}(v_n) \\
& +2d_{G_1}(u_i)d_{G_1}(u_m)d_{G_2}(v_l)d_{G_2}(v_n)-4d_{G_1}(u_i)d_{G_2}(v_l) \\
& -4d_{G_1}(u_m)-4d_{G_2}(v_n)-4d_{G_1}(u_m)d_{G_2}(v_n)\Big] \\
= & \sum_{i,m=0,i\neq m}^{p_1-1} 1 \sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l,v_n)[d_{G_2}(v_l)+d_{G_2}(v_n)-2]^2 \\
& +\sum_{i,m=0,i\neq m}^{p_1-1}[d_{G_1}(u_i)+d_{G_1}(u_m)]^2 \sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l,v_n) \\
& +2\sum_{i,m=0,i\neq m}^{p_1-1}[d_{G_1}(u_i)-d_{G_1}(u_m)] \sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l,v_n)d_{G_2}(v_n) \\
& -2\sum_{i,m=0,i\neq m}^{p_1-1}[d_{G_1}(u_i)-d_{G_1}(u_m)] \sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l,v_n)d_{G_2}(v_l) \\
& +2\sum_{i,m=0,i\neq m}^{p_1-1} d_{G_1}(u_i)d_{G_1}(u_m) \sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l,v_n)[d_{G_2}(v_l)+d_{G_2}(v_n)] \\
& +2\sum_{i,m=0,i\neq m}^{p_1-1}[d_{G_1}(u_i)+d_{G_1}(u_m)] \sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l,v_n)[d_{G_2}(v_l)d_{G_2}(v_n)] \\
& +\sum_{i,m=0,i\neq m}^{p_1-1} d^2_{G_1}(u_i) \sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l,v_n)d^2_{G_2}(v_l) \\
& +\sum_{i,m=0,i\neq m}^{p_1-1} d^2_{G_1}(u_m) \sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l,v_n)d^2_{G_2}(v_n) \\
& +2\sum_{i,m=0,i\neq m}^{p_1-1} d_{G_1}(u_i)d_{G_1}(u_m) \sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l,v_n)[d_{G_2}(v_l)d_{G_2}(v_n)] \\
& +2\sum_{i,m=0,i\neq m}^{p_1-1} d_{G_1}(u_m) \sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l,v_n)d^2_{G_2}(v_n) \\
& +2\sum_{i,m=0,i\neq m}^{p_1-1} d^2_{G_1}(u_i) \sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l,v_n)d_{G_2}(v_l) \\
& +2\sum_{i,m=0,i\neq m}^{p_1-1} d_{G_1}(u_i) \sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l,v_n)d^2_{G_2}(v_l)
\end{aligned}
$$

$$
\begin{aligned}
&+2\sum_{i,m=0,i\neq m}^{p_1-1} d^2_{G_1}(u_m)\sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l,v_n)d_{G_2}(v_n)\\
&-4\sum_{i,m=0,i\neq m}^{p_1-1}[d_{G_1}(u_i)+d_{G_1}(u_m)]\sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l,v_n)\\
=\;& 2p_1(p_1-1)DEM_1(G_2)+4W(G_2)[(p_1-1)M_1(G_1)+4q_1^2-M_1(G_2)\\
&-8q_1(p_1-1)]+[4DD(G_2)+4Gut(G_2)][4q_1^2-M_1(G_1)]\\
&+8q_1(p_1-1)[2Gut(G_2)+DF(G_2)]+2(p_1-1)M_1(G_1)[DF(G_2)+2DD(G_2)]
\end{aligned}
$$

*Using* (2), (3) *and* (4), *in* (1) *we get,*

$$
\begin{aligned}
2\times DE(G_1\boxtimes G_2) =\;& 2p_2DEM_1(G_1)+8q_2DHM(G_1)+8M_1(G_2)DD(G_1)\\
&+2M_1(G_2)DHM(G_1)-32q_2W(G_1)+8M_1(G_2)W(G_1)\\
&+2p_1DEM_1(G_2)+8q_1DHM(G_2)+8M_1(G_1)DD(G_2)\\
&+2M_1(G_1)DHM(G_2)-32q_1W(G_2)+8M_1(G_1)W(G_2)\\
&+2p_1(p_1-1)DEM_1(G_2)+[4DD(G_2)+4Gut(G_2)][4q_1^2-M_1(G_1)]\\
&+4W(G_2)[(p_1-1)M_1(G_1)+4q_1^2-M_1(G_2)-8q_1(p_1-1)]\\
&+8q_1(p_1-1)[2Gut(G_2)+DF(G_2)]\\
&+2(p_1-1)M_1(G_1)[DF(G_2)+2DD(G_2)]
\end{aligned}
$$

## 2.3 Distance Based Reformulated First Zagreb Index of Corona Product of Graphs

In this section, we compute the distance based reformulated first Zagreb index of corona product $G_1\odot G_2$ of the graphs $G_1$ and $G_2$.

Let $V(G_1)=\{u_0,u_1,...,u_{p_1-1}\}$ and $V(G_2)=\{v_0,v_1,...,v_{p_2-1}\}$. For $0\leq i\leq p_1-1$, denote by $G_2^i$ the $i^{th}$ copy of $G_2$ joined to the vertex $u_i$ and $V(G_2^i)=\{v_{i0},v_{i1},...,v_{i(p_2-1)}\}$.

**Lemma 2.7.** [12] *The degree of* $w\in V(G_1\odot G_2)$ *is*

$$
d_{G_1\odot G_2}(w)=\begin{cases} d_{G_1}(w)+p_2 & if\ w\in V(G_1)\\ d_{G_1}(w)+p_2 & if\ w\in V(G_1^i)\ for\ some\ 0\leq i\leq p_1-1.\end{cases}
$$

**Lemma 2.8.** *Let* $G_1$ *be arbitrary graph.Let* $G_2^i$ *be the* $i^{th}$ *copy of* $G_2$ *in* $d_{G_1\odot G_2}$ *and let* $V(G_2^i)=\{v_{i0},v_{i1},...,v_{i(p_2-1)}\}$. *Then*

$d_{(G_1\odot G_2)}(u_i,u_m)=d_{G_1}(u_i,u_m),\ if\ 0\leq i,m\leq p_1-1,$

$$d_{(G_1\odot G_2)}(u_i, v_{mn}) = d_{G_1}(u_i, u_m) + 1,\ if\ 0 \le i, m \le p_1 - 1, 0 \le n \le p_2 - 1,$$

$$d_{G_1\odot G_2}(v_{il}, v_{mn}) = \begin{cases} d_{G_1}(u_i, u_m) + 2, & if\ i \ne m, \\ 1, & if\ i = m,\ and\ v_l v_n \in E(G_2), \\ 2, & if\ i = m,\ and\ v_l v_n \notin E(G_2). \end{cases}$$

**Theorem 2.9.** *Let $G_i, i = 1, 2$, be a $(p_i, q_i)$- graph. Then*

$$\begin{aligned} 2 \times DF(G_1 \odot G_2) &= 2DE(G_1) + 4p_2^2 W(G_1) + 4p_2 DD(G_1) + 2p_1\Big[2p_2 M_1(G_2) + 8p_2 q_2 \\ &\quad +2p_2(p_2 - 1) - F(G_2) - 4M_1(G_2) - 10q_2\Big] + 2p_2 DE(G_1) \\ &\quad +4p_2^2 DD(G_1) + 4W(G_1)[p_2(p_2^2 + 1)M_1(G_2) + 4q_2] \\ &\quad +2p_1 p_2[M_1(G_1) + 4q_1 p_2] + 2p_1^2[M_1(G_2) + 4q_2 + p_2(p_2^2 + 1)] \\ &\quad +4[W(G_1) + p_1(p_1 - 1)][p_2 M_1(G_2) + 4p_2 q_2 + p_2^2] \end{aligned}$$

**Proof***: Let $G = G_1 \odot G_2$. Then,*

$$\begin{aligned} 2 \times DEM_1(G) &= \sum_{u,v \in V(G)} d_G(u, v)[d_G(u) + d_G(v) - 2]^2 \\ &= \sum_{i,m=0, i\ne m}^{p_1-1} d_G(u_i, u_m)[d_G(u_i) + d_G(u_m) - 2]^2 \\ &\quad + \sum_{i=0}^{p_1-1} \sum_{l,n=0, l\ne m}^{p_2-1} d_G(v_{il}, v_{in})[d_G(v_{il}) + d_G(v_{in}) - 2]^2 \\ &\quad +2\sum_{m=0}^{p_1-1} \sum_{i=0}^{p_1-1} \sum_{n=0}^{p_2-1} [d_G(u_i, v_{mn})[d_G(u_i) + d_G(v_{mn}) - 2]^2 \\ &\quad + \sum_{i,m=0, i\ne m}^{p_1-1} \sum_{l,n=0}^{p_2-1} d_G(v_{il}, v_{mn})[d_G(v_{il}) + d_G(v_{mn}) - 2]^2 \end{aligned}$$

$$= A_1 + A_2 + A_3 + A_4 \ where\ A_1,\ A_2,\ A_3,\ A_4 \qquad (1)$$

*are the sums of the above terms in order. We calculate $A_1,\ A_2,\ A_3$ and $A_4$ semarately. First we commute $A_1$*

$$\begin{aligned} A_1 &= \sum_{i,m=0, i\ne m}^{p_1-1} d_G(u_i, u_m)[d_{G_1}(u_i) + d_{G_1}(u_m) - 2]^2 \\ &= \sum_{i,m=0, i\ne m}^{p_1-1} d_{G_1}(u_i, u_m)\Big[d_{G_1}(u_i) + p_2 + d_{G_1}(u_m) + p_2 - 2\Big]^2 \\ &= \sum_{i,m=0, i\ne m}^{p_1-1} d_{G_1}(u_i, u_m)\Big[d_{G_1}^2(u_i) + d_{G_1}^2(u_m) + 4p_2^2 + 4 + 2d_{G_1}(u_i)d_{G_1}(u_m) \end{aligned}$$

$$
\begin{aligned}
&+4p_2d_{G_1}(u_i) - 4d_{G_1}(u_i) + 4p_2d_{G_1}(u_m) - 4d_{G_1}(u_m) - 8p_2\Big] \\
= &\sum_{i,m=0,i\neq m}^{p_1-1} d_{G_1}(u_i,u_m)\Big[d_{G_1}(u_i)+d_{G_1}(u_m)-2\Big]^2 + 4n_2^2\sum_{i,m=0,i\neq m}^{p_1-1} d_{G_1}(u_i,u_m) \\
&+4p_2\sum_{i,m=0,i\neq m}^{p_1-1} d_{G_1}(u_i,u_m)\Big[d_{G_1}(u_i)+d_{G_1}(u_m)\Big] - 8p_2\sum_{i,m=0,i\neq m}^{p_1-1} d_{G_1}(u_i,u_m) \\
= &\ 2[DEM_1(G_1) + 4p_2^2W(G_1) + 4p_2DD(G_1) - 8p_2W(G_1)]
\end{aligned}
$$

*Next we commute* $A_2$

$$
\begin{aligned}
A_2 = &\sum_{i=0}^{p_1-1}\sum_{l,n=0,l\neq n}^{p_2-1} d_G(v_{il},v_{in})\Big[d_G(v_{il})+d_G(v_{in})-2\Big]^2 \\
= &\sum_{i=0}^{p_1-1}\Bigg[\sum_{l,n=0,l\neq n,\ v_lv_n\in E(G_2)}^{p_2-1} d_G(v_{il},v_{in})\Big[d_G(v_{il})+d_G(v_{in})-2\Big]^2 \\
&+\sum_{l,n=0,l\neq n,\ v_lv_n\notin E(G_2)}^{p_2-1} d_G(v_{il},v_{in})\Big[d_G(v_{il})+d_G(v_{in})-2\Big]^2\Bigg] \\
= &\sum_{i=0}^{p_1-1}\Bigg[\sum_{l,n=0,l\neq n,\ v_lv_n\in E(G_2)}^{p_2-1} \Big[d_{G_2}(v_l)+1+d_{G_2}(v_n)+1-2\Big]^2 \\
&+\sum_{l,n=0,l\neq n,\ v_lv_n\notin E(G_2)}^{p_2-1} 2\Big[d_{G_2}(v_l)+1+d_{G_2}(v_n)+1-2\Big]^2\Bigg] \\
= &\sum_{i=0}^{p_1-1}\Bigg[\sum_{l,n=0,l\neq n,\ v_lv_n\in E(G_2)}^{p_2-1} \Big[(d^2_{G_2}(v_l)+d^2_{G_2}(v_l)+2d_{G_2}(v_l)d_{G_2}(v_n)\Big] \\
&+\sum_{l,n=0,l\neq n,\ v_lv_n\notin E(G_2)}^{p_2-1} 2\Big[(d^2_{G_2}(v_l)+d^2_{G_2}(v_l)+2d_{G_2}(v_l)d_{G_2}(v_n)\Big] \\
= &\sum_{i=0}^{p_1-1}\Bigg[\Big\{\sum_{l,n=0,l\neq n,\ v_lv_n\in E(G_2)}^{p_2-1} \Big[(d^2_{G_2}(v_l)+d^2_{G_2}(v_l)+2d_{G_2}(v_l)d_{G_2}(v_n)\Big] \\
&+\sum_{l,n=0,l\neq n,\ v_lv_n\notin E(G_2)}^{p_2-1} \Big[(d^2_{G_2}(v_l)+d^2_{G_2}(v_l)+2d_{G_2}(v_l)d_{G_2}(v_n)\Big] \\
&+\sum_{l,n=0,l\neq n,\ v_lv_n\notin E(G_2)}^{p_2-1} \Big[(d^2_{G_2}(v_l)+d^2_{G_2}(v_l)+2d_{G_2}(v_l)d_{G_2}(v_n\Big] \\
= &\sum_{i=0}^{p_1-1}\Bigg[\sum_{l,n=0,l\neq n,}^{n_2-1} \Big[(d^2_{G_2}(v_l)+d^2_{G_2}(v_l)+2d_{G_2}(v_l)d_{G_2}(v_n)\Big] \\
&+\Big\{\sum_{v_lv_n\in E(G_2)}^{p_2-1} \Big[(d^2_{G_2}(v_l)+d^2_{G_2}(v_l)+2d_{G_2}(v_l)d_{G_2}(v_n)2\Big] \\
&+\sum_{v_lv_n\notin E(G_2)}^{p_2-1} \Big[(d^2_{G_2}(v_l)+d^2_{G_2}(v_l)+2d_{G_2}(v_l)d_{G_2}(v_n)\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{l,n=0,l\neq n,\ v_lv_n\in E(G_2)}^{p_2-1} \Big[(d^2_{G_2}(v_l)+d^2_{G_2}(v_l)+2d_{G_2}(v_l)d_{G_2}(v_n)\Big]\\
&= \sum_{i=0}^{p_1-1}\Big[2\sum_{l,n=0,l\neq n,}^{p_2-1}\Big[(d^2_{G_2}(v_l)+d^2_{G_2}(v_l)+2\Big(d_{G_2}(v_l)+d_{G_2}(v_n)\Big)+2\Big]\\
&\quad -2\sum_{l,n=0,l\neq n,\ v_lv_n\in E(G_2)}^{p_2-1}\Big[(d^2_{G_2}(v_l)+d^2_{G_2}(v_l)+2\Big(d_{G_2}(v_l)+d_{G_2}(v_n)\Big)+2\Big]\Big]\\
&= \sum_{i=0}^{p_1-1}\Big[4(n_2-1)M_1(G_2)+4(4q_2^2-M_1(G_2))-2F(G_2)-4M_2(G_2)\Big]\\
&= 2p_1\Big[2p_2M_1(G_2)+8q_2^2-F(G_2)-4M_1(G_2)-2M_2(G_2)\Big]
\end{aligned}
$$

*Next we compute $A_3$.*

$$
\begin{aligned}
A_3 &= 2\sum_{m=0}^{p_1-1}\sum_{i=0}^{p_1-1}\sum_{n=0}^{p_2-1} d_G(u_i,v_{mn})\Big[d_G(u_i)+d_G(u_{mn})-2\Big]^2\\
&= 2\sum_{m=0}^{p_1-1}\sum_{i=0}^{p_1-1}\sum_{n=0}^{p_2-1}[d_{G_1}(u_i,u_m)+1]\Big[d_{G_1}(u_i)+p_2+d_{G_2}(v_n)+1-2\Big]^2\\
&= 2\sum_{m=0}^{p_1-1}\sum_{i=0}^{p_1-1}\sum_{n=0}^{p_2-1}\Big[d_{G_1}(u_i,u_m)+1\Big]\Big[d^2_{G_1}(u_i)+p_2^2+d^2_{G_2}(v_n)+1+2p_2d_{G_1}(u_i)\\
&\quad +2d_{G_1}(u_i)d_{G_2}(v_n)-2d_{G_1}(u_i)+2p_2d_{G_2}(v_n)-2p_2-2d_{G_2}(v_n)\Big]\\
&= 2\sum_{m=0}^{p_1-1}\sum_{i=0}^{p_1-1}\sum_{n=0}^{p_2-1}\Big[d_{G_1}(u_i,u_m)\Big[d^2_{G_2}(u_i)+d^2_{G_1}(v_n)+p_2^2+1+2p_2d_{G_1}(u_i)\\
&\quad +2d_{G_1}(u_i)d_{G_2}(v_n)-2d_{G_1}(u_i)+2p_2d_{G_2}(v_n)-2p_2-2d_{G_2}(v_n)\Big]\\
&\quad +\Big[d^2_{G_1}(u_i)+d^2_{G_2}(v_n)+p_2^2+1+2p_2d_{G_1}(u_i)\\
&\quad +2d_{G_1}(u_i)d_{G_2}(v_n)-2d_{G_1}(u_i)+2p_2d_{G_2}(v_n)-2n_2-2d_{G_2}(v_n)\Big]\Big]\\
&= 2\sum_{m=0}^{p_1-1}\sum_{i=0}^{p_1-1}\Big[p_2d_{G_1}(u_i,u_m)d^2_{G_1}(u_i)+M_1(G_2)d_{G_1}(u_i,u_m)+p_2^3d_{G_1}(u_i,u_m)\\
&\quad +p_2d_{G_1}(u_i,u_m)+2p_2^2d_{G_1}(u_i,u_m)d_{G_1}(u_i)+4q_2d_{G_1}(u_i,u_m)d_{G_1}(u_i)\\
&\quad -2p_2d_{G_1}(u_i,u_m)d_{G_1}(u_i)+4q_2p_2d_{G_1}(u_i,u_m)-2p_2^2d_{G_1}(u_i,u_m)\\
&\quad -4q_2d_{G_1}(u_i,u_m)+p_2d^2_{G_1}(u_i)+M_1(G_2)+p_2^3+p_2\\
&\quad +2p_2^2d_{G_1}(u_i)+4q_2d_{G_1}(u_i)-2p_2d_{G_1}(u_i)+4q_2p_2-2p_2^2-4q_2\Big]\\
&= 2\Big[p_2\sum_{m=0}^{p_1-1}\sum_{i=0}^{p_1-1}p_2d_{G_1}(u_i,u_m)d^2_{G_1}(u_i)+M_1(G_2)\sum_{m=0}^{p_1-1}\sum_{i=0}^{p_1-1}d_{G_1}(u_i,u_m)\\
&\quad +p_2^3\sum_{m=0}^{p_1-1}\sum_{i=0}^{p_1-1}d_{G_1}(u_i,u_m)+p_2\sum_{m=0}^{p_1-1}\sum_{i=0}^{p_1-1}d_{G_1}(u_i,u_m)
\end{aligned}
$$

$$
\begin{aligned}
&+2p_2^2\sum_{m=0}^{p_1-1}\sum_{i=0}^{p_1-1} d_{G_1}(u_i,u_m)d_{G_1}(u_i) + 4q_2\sum_{m=0}^{p_1-1}\sum_{i=0}^{p_1-1} d_{G_1}(u_i,u_m)d_{G_1}(u_i)\\
&-2p_2\sum_{m=0}^{p_1-1}\sum_{i=0}^{p_1-1} d_{G_1}(u_i,u_m)d_{G_1}(u_i) - 4q_2p_2\sum_{m=0}^{p_1-1}\sum_{i=0}^{p_1-1} d_{G_1}(u_i,u_m)\\
&-2p_2^2\sum_{m=0}^{p_1-1}\sum_{i=0}^{p_1-1} d_{G_1}(u_i,u_m) + 4q_2\sum_{m=0}^{p_1-1}\sum_{i=0}^{p_1-1} d_{G_1}(u_i,u_m)\\
&+p_2\sum_{m=0}^{p_1-1}\sum_{i=0}^{p_1-1} d_{G_1}^2(u_i) + M_1(G_2)\sum_{m=0}^{p_1-1}\sum_{i=0}^{p_1-1} 1\\
&+p_2^3\sum_{m=0}^{p_1-1}\sum_{i=0}^{p_1-1} 1 + p_2\sum_{m=0}^{p_1-1}\sum_{i=0}^{p_1-1} 1\\
&+2p_2^2\sum_{m=0}^{p_1-1}\sum_{i=0}^{p_1-1} d_{G_1}(u_i) + 4q_2\sum_{m=0}^{p_1-1}\sum_{i=0}^{p_1-1} d_{G_1}(u_i)\\
&-2p_2\sum_{m=0}^{p_1-1}\sum_{i=0}^{p_1-1} d_{G_1}(u_i) + 4q_2p_2\sum_{m=0}^{n_1-1}\sum_{i=0}^{p_1-1} 1\\
&-2p_2^2\sum_{m=0}^{p_1-1}\sum_{i=0}^{p_1-1} 1 - 4q_2\sum_{m=0}^{p_1-1}\sum_{i=0}^{p_1-1} 1\Big]\\
=\;& 2\Big[2W(G_1)\Big[M_1(G_2) + p_2(p_2^2+1) + 4q_2p_2 - 2p_2^2 - 4q_2\Big] + 2DD(G_1)[p_2^2 + 2q_2 - p_2]\\
&+p_2DF(G_1) + p_1p_2[M_1(G_1) + p_1(p_2^2+1) + 4q_1p_2 - 2p_2p_1 + 4p_1q_2 - 4q_1]\\
&+p_1^2[M_1(G_2) - 4q_2] + 8q_1q_2p_1]\Big]
\end{aligned}
$$

*Finally we calculate* $A_4$.

$$
\begin{aligned}
A_4 &= \sum_{i,m=0,i\neq m}^{p_1-1}\sum_{l,n=0}^{p_2-1} d_G(v_{il},v_{mn})[d_G(v_{il}) + d_G(v_{mn}) - 2]^2\\
&= \sum_{i,m=0,i\neq m}^{p_1-1}\sum_{l,n=0}^{p_2-1} [d_{G_1}(u_i,u_m) + 2][d_G(v_l) + 1 + d_G(v_n) + 1 - 2]^2\\
&= \sum_{i,m=0,i\neq m}^{p_1-1} [d_{G_1}(u_i,u_m) + 2]\;\sum_{l,n=0}^{p_2-1}\Big[d_{G_2}^2(v_l) + d_{G_2}^2(v_n)\\
&\quad +d_{G_2}(v_l)d_{G_2}(v_n)\Big]\\
&= \sum_{i,m=0,i\neq m}^{p_1-1} [d_{G_1}(u_i,u_m) + 2]\;\; 4\sum_{l,n=0}^{p_2-1} d_{G_2}^2(v_l)\\
&= 8p_2M_1(G_2)[W(G_1) + p_1(p_1-1)]
\end{aligned}
$$

*Using* (6), (7), (8) *and* (9), *in* (5) *we get*

$$
2\times DE(G_1\odot G_2) = 2DF(G_1) + 4p_2^2W(G_1) + 4p_2DD(G_1) + 2p_1\Big[2p_2M_1(G_2) + 8p_2q_2
$$

$$+2p_2(p_2-1)-F(G_2)-4M_1(G_2)-10q_2\Big]+2p_2DF(G_1)$$
$$+4p_2^2DD(G_1)+4W(G_1)[p_2(p_2^2+1)M_1(G_2)+4q_2]$$
$$+2p_1p_2[M_1(G_1)+4q_1p_2]+2p_1^2[M_1(G_2)+4q_2+p_2(p_2^2+1)]$$
$$+4[W(G_1)+p_1(p_1-1)][p_2M_1(G_2)+4p_2q_2+p_2^2]$$

### 2.4 Distance Based Reformulated First Zagreb Index of Tensor Product of Graphs

In this section, we compute the distance based reformulated first Zagreb index of tensor product $G_1 \times G_2$ of the graphs $G_1$ and $G_2$.

**Lemma 2.10.** [7]

$$\begin{aligned}(i)|V(G_1\times G_2)| &= |V(G_1)||V(G_2)|\\ (ii)|E(G_1\times G_2)| &= 2|E(G_1)||E(G_2)|\\ (iii)d_{G_1\times G_2}(u_i,v_l) &= d_{G_1}(u_i)d_{G_2}(v_l)\end{aligned}$$

**Lemma 2.11.** *Let $w_{il}=(u_i,v_l)$ and $w_{mn}=(u_m,v_n)$ be in $V(G_1\times G_2)$. Then the distance between $w_{il}$ and $w_{mn}$ is*

$$d_{G_1\times G_2}(w_{il},w_{mn})=\begin{cases}d_{G_2}(v_l,v_n), & \text{if, } i\neq m, l\neq n\\ 0, & \text{otherwise}\end{cases}$$

**Theorem 2.12.** *Let $G_i, i=1,2$, be a $(p_i,q_i)$- graph. Then*

$$\begin{aligned}DEM_1(G) &= (p_1-1)M_1(G_1)DF(G_2)+4p_1(p_1-1)W(G_2)+2[4q_1^2-M_1(G_1)]Gut(G_2)\\ &- 8(p_1-1)m_1DD(G_2)\end{aligned}$$

**Proof***: Let $G_1\times G_2=G$. Then*

$$\begin{aligned}2\times DEM_1(G) &= \sum_{w_{il},w_{mn}\in V(G)} d_G(w_{il},w_{mn})\Big[d_G(w_{il})+d_G(w_{mn})-2\Big]^2\\ &= \sum_{i,m=0,i\neq m}^{p_1-1}\ \sum_{l,n=0,l\neq n}^{p_2-1} d_G(w_{il},w_{mn})\Big[d_G(w_{il})+d_G(w_{mn})-2\Big]^2\\ &= \sum_{i,m=0,i\neq m}^{p_1-1}\ \sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l,v_n)\Big[d_{G_1}(u_i)d_{G_2}(v_l)+d_{G_1}(u_m)d_{G_2}(v_n)-2\Big]^2\\ &= \sum_{i,m=0,i\neq m}^{p_1-1}\ \sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l,v_n)\Big[d_{G_1}^2(u_i)d_{G_2}^2(v_l)+d_{G_1}^2(u_m)d_{G_2}^2(v_n)+4\\ &\quad +2d_{G_1}(u_i)d_{G_1}(u_m)d_{G_2}(v_l)d_{G_2}(v_n)-4d_{G_1}(u_i)d_{G_2}(v_l)-4d_{G_1}(u_m)d_{G_2}(v_n)\Big]\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i,m=0,i\neq m}^{p_1-1} d^2_{G_1}(u_i) \sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l,v_n) d^2_{G_2}(v_l) \\
&\quad + \sum_{i,m=0,i\neq m}^{p_1-1} d^2_{G_1}(u_m) \sum_{l,n=0,l\neq n}^{n_2-1} d_{G_2}(v_l,v_n) d^2_{G_2}(v_n) \\
&\quad + 4 \sum_{i,m=0,i\neq m}^{p_1-1} 1 \sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l,v_n) \\
&\quad + 2 \sum_{i,m=0,i\neq m}^{p_1-1} d_{G_1}(u_i) d_{G_1}(u_m) \sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l,v_n) d_{G_2}(v_l) d_{G_2}(v_n) \\
&\quad - 4 \sum_{i,m=0,i\neq m}^{p_1-1} d_{G_1}(u_i) \sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l,v_n) d_{G_2}(v_l) \\
&\quad - 4 \sum_{i,m=0,i\neq m}^{p_1-1} d_{G_1}(u_m) \sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l,v_n) d_{G_2}(v_n)
\end{aligned}
$$

$$
\begin{aligned}
DEM_1(G) &= (p_1-1) M_1(G_1) DF(G_2) + 4p_1(p_1-1) W(G_2) + 2[4q_1^2 - M_1(G_1)] Gut(G_2) \\
&\quad - 8(p_1-1) m_1 DD(G_2)
\end{aligned}
$$

## References


[1] M. Aruvi, V. Piramanantham and R. Muruganandam, Distance version of Hyper Zagreb index of some graph operations, *J. Int. Math. Virtual Inst.*, Vol. 10(1) (2020), 1–40.

[2] R. Balakrishnan, The energy of a graph, *Linear Algebra and its Applications*, Vol. 573 (2019).

[3] A. A. Dobrynin and A. A. Kochetova, Degree distance of a graph, A degree analogue of the Wiener index, *J. Chem. Inf. Comput. Sci.*, Vol. 34 (1994), 1082–1086.

[4] B. Furtula and I. Gutman, A forgotten topological index, *J. Math. Chem.*, Vol. 53(4) (2015), 1184–1190.

[5] I. Gutman, Selected properties of the Schultz molecular topological index, *J. Chem. Inf. Comput. Sci.*, Vol. 34 (1994), 1087–1089.

[6] I. Gutman and N. Trinajstic, Graph theory and molecular orbitals, Total $\pi$-electron energy of alternant hydrocarbons, *Chem. Phys. Lett.*, Vol. 17 (1972), 535–538.

[7] R. Hammack, W. Imrich and S. Klazar, Handbook of product graphs, CRC Press, New York, (2011).

[8] A. Milicevic, S. Nikolic and N. Trinajstic, On reformulated Zagreb indices, *Discrete Applied Mathematics*, Vol. 160 (2012), 204–209.

[9] R. Muruganandam, R. S. Manikandan and M. Aruvi, Distance version of $F$-index of some graph operations, *International Journal of Applied Graph Theory*, Vol. 3(1) (2019), 01–36.

[10] N. De, S. K. M. D. Abu Nayeem and A. Pal, F-index of some graph operations, *Discrete Mathematics, Algorithms and Applications*, doi:10.1142/S1793830916500257.

[11] N. De, S. K. M. D. Abu Nayeem and A. Pal, Reformulated first Zagreb index of some graph operations, *Mathematics*, Vol. 3 (2015), 945–960, doi:10.3390/math3040945.

[12] G. H. Shirdel, H. Rezapour and A. M. Sayadi, The Hyper-Zagreb index of graph operations, *Iranian Journal of Mathematical Chemistry*, Vol. 2 (2013), 213–220.

[13] N. Trinajstic, Chemical graph theory, CRC Press, Boca Raton, FL, (1983).

[14] H. Wiener, Structural determination of the paraffin boiling points, *J. Amer. Chem. Soc.*, Vol. 69 (1947), 17–20.