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Graham's Pebbling Conjecture for Book Graphs

S. Kither iammal
Department of Mathematics
Jayaraj Annapackiam College for women (Autonomous), Periyakulam
Tamilnadu, India
cathsat86@gmail.com
https://orcid.org/0000-0003-3553-0848

A. Lourdusamy
Department of Mathematics
St. Xavier's College (Autonomous), Palayamkottai-627002
Tamil Nadu, India
lourdusamy15@gmail.com
https://orcid.org/0000-0001-5961-358X

I. Dhivviyanandam
Department of Mathematics
North Bengal St. Xavier's College
Rajganj, west Bengal, India
divyanasj@gmail.com
https://orcid.org/0000-0002-3805-6638

Abstract

Consider a configuration of pebbles on the vertices of a connected graph. A pebbling move is to remove two pebbles from a vertex and to place one pebble at the neighbouring vertex of the vertex from which the pebbles are removed. The pebbling number of a vertex v in a graph G is the smallest number $f(G,v)$ such that for every placement of $f(G,v)$ pebbles, it is possible to move a pebble to v by a sequence of pebbling moves. The pebbling number of G is the smallest number, $f(G)$, such that from any distribution of $f(G)$ pebbles, it is possible to move a pebble to any specified target vertex by a sequence of pebbling moves. Thus, $f(G)$ is the maximum value of $f(G,v)$ overall vertices v. For any connected graph G and H, Graham conjectured that $f(G\times H)\leq f(G)f(H)$. In this paper, we prove that Graham's pebbling conjecture holds for book graphs.

Key words: pebbling moves, pebbling number, 2-pebbling property, Graham's conjecture, book graph

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* Corresponding Author: $I. Dhivviyanandam$
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1 Introduction

Pebbling was introduced by Lagarias and Saks. F.R.K. Chung [2] developed the fundamental concepts on pebbling and published the results in the year 1989. Hulbert published a survey article on pebbling which enhanced the interest of researchers to explore further on graph pebbling [4].

Through out this paper G will denote a simple connected graph. Let us denote G's vertex and edge sets as $V(G)$ and $E(G)$, respectively. Consider a configuration of pebbles on the vertices of a connected graph. A pebbling move is to remove two pebbles from a vertex and to place one pebble at the neighbouring vertex of the vertex from which the pebbles are removed. Chung [2] defined the pebbling number of a connected graph G, which we denote $f(G)$. The pebbling number of a vertex v in a graph G is the smallest number $f(G, v)$ that allows us to shift a pebble to v using a sequence of pebbling transformation, regardless of where these pebbles are located on G's vertices. The pebbling number, $f(G)$, of a graph G is the maximum $f(G, v)$ over all the vertices of a graph.

Chung [2] defined the 2-pebbling property and Wang [10] extended Chung's definition to the odd 2-pebbling property. Given a distribution of pebbles on G, let p be the number of pebbles in that distribution, let q be the number of occupied vertices (vertices with at least one pebble), and let r be the number of vertices with the odd number of pebbles. We say that G satisfies the 2-pebbling property (respectively, the odd 2-pebbling property), if it is possible to move two pebbles to any specified target vertex whenever p and q satisfy the inequality $p + q > 2f(G)$ (respectively, whenever p and r satisfy $p + r > 2f(G)$).

Lourdusamy et al [5], [6], [7], [8] studied that the n-cube graphs, the complete graphs, the even cycle graphs, the complete r-partite graphs, the star graphs, the wheel graphs, and the fan graphs have the 2-pebbling property. The following conjecture [2], by Ronald Graham, suggests a constraint on the pebbling number of the product of two graphs.

Conjecture 1.1. The pebbling number of $G \times H$ is $f(G \times H) \leq f(G) f(H)$.

2 Preliminaries

For the basic definitions in graph theory, the reader can read [1].

Definition 2.1. [3] Let $V(G \times H) = V(G) \times V(H)$. The edge set of $G \times H$ is define as $E(G \times H) = \{((x, y), (x', y')) : x = x' \ and \ (y, y') \in E(H) \ or \ (x, x') \in E(G) \ and \ y = y'\} \ \ and \ (y, y') \in E(H) \ or \ (x, x') \in E(G) \ and \ y = y'\}$. The graph $G \times H$ is called the Cartesian product of G and H.

Definition 2.2. [12] The star graph S_m of order m is a tree on m nodes with one node having vertex degree $m-1$ and the other $m-1$ having vertex degree 1. The star graph S_m is therefore isomorphic to the complete bipartite graph $K_{(1,n-1)}$.

Definition 2.3. [11] The m book is defined as the graph Cartesian product $B_m = S_{m+1} \times P_2$ where S_m is a star graph and P_2 is the path graph on two nodes.

Theorem 2.4. [9] The pebbling number of B_m is $f(B_m) = 2m + 4$.

Theorem 2.5. [9] The Book graph B_m satisfies the 2-pebbling property.

Notation 2.6. Let $p\,(v)$ be the number of pebbles on v. Let $p^{\sim}(v)$ be the number of pebbles placed on the vertices other than v. Further, we denote P_i as i^{th} path and $P_i^{\sim}$ be the vertices which are not on i^{th} path P_i, where i is the positive integer. Consider the adjacent vertices x_i and x_l. The notation $(x_i)\underset{\rightarrow}{t}(x_l)$ refers to taking off at least $2t$ pebbles from (x_i) and placing at least t pebbles on (x_l). We use β to denote the destination vertex.

Definition 2.7. [3] A transmitting subgraph of a graph G is a path $v_0, v_1, v_2 \cdots, v_n$ in which one pebble is transmitted from v_0 to v_n with the distribution of at least two pebbles in v_0 and at least one pebble on each of the other vertices in the path, except possibly v_n. With this distribution of pebbles one can transmit a pebble from v_0 to v_n.

Remark 2.8. From G, we select a target vertex β. We can easily shift a pebble to β if $p(\beta) = 1$ or $p\,(s) \geq 2$, where $s \in E\,(G)$. When β is the target vertex, we always assume that $p\,(\beta) = 0$ and $p\,(s) \leq 1$.

3 Graham's pebbling conjecture for book graphs

Theorem 3.1. If H satisfies the 2-pebbling property then for $n \geq 2$, $f(B_n \times H) \leq (2n + 4)f(H)$

Proof: Let $G = B_n$ and $H = B_m$. Let $V(B_n) = \{u_1,\ u_2,\ v_i\}$ where $i = \{1,\ 2,\ \cdots,\ 2n\}$ and $V(B_m) = \{x_1,\ x_2,\ y_j\}$ where $j = \{1,\ 2,\ \cdots,\ 2m\}$. We prove the Graham pebbling conjecture. Let D be any distribution of $(2n+4)f(B_m)$ pebbles on the vertices of $B_n \times B_m$.

Case 1. Let $(u_1,\ x_k)$ or $(u_1,\ y_j)$ be the target vertex where $1 \leq k \leq 2$ and $1 \leq j \leq 2m$. Without loss of generality, let $(u_1,\ x_1)$ be the destination vertex. We take n copies of B_m. Denote $\{u_1\} \times H$, $\{u_2\} \times H$, $\{v_1\} \times H,\ \cdots,\ \{v_{2n}\} \times H$ respectively as $H_1,\ H_2,\ \cdots,\ H_{2m}$, $H_{2m+1},\ H_{2m+2}$. Let α_1 be the number of pebbles on the vertices of $\{u_1\} \times H_1$, Let α_2 be the number of pebbles on the vertices of $\{u_2\} \times H_2$, Let β_j be the number of pebbles on the vertices of $\{v_j\} \times H_j$, where $1 \leq j \leq 2m$. Let c_1 be the number of occupied vertices on $\{u_1\} \times H_1$, c_2 be the number of occupied vertices on $\{u_2\} \times H_2$, d_j be the number of occupied vertices on $\{v_j\} \times H_j$, where $1 \leq j \leq 2m$.

If $\alpha_1 \geq f(H)$, then we can transfer 1 pebbles to $(u_1,\ x_1)$. So let $\alpha_1 < f(H)$. If $\alpha_2 \geq 2f(H)$, then we can transfer 2 pebbles to $(u_2,\ x_1)$ and further we can move 1 pebble to $(u_1,\ x_1)$. Otherwise if $\beta_k \geq 4f(H)$ where $k = \{1,\ 3,\ \cdots,\ 2m+1\}$ we can reach the target. Or any one $\beta_s \geq 2f(H)$ where $s = \{2,\ 4,\ \cdots,\ 2m+2\}$ we can transfer 4 pebbles to $(v_s,\ x_s)$ and further we can transfer 2 pebbles to $(u_2,\ x_1)$ so we reach the target. Suppose $\alpha_1 < f(H)$, $\frac{\beta_k - d_k}{4} < f(H)$, $\frac{\beta_s - d_s}{4} < 2f(H)$ then we have one of the following:

$$\alpha_1 + \frac{\alpha_2 - c_2}{2} + \sum_{k=1,\ 3,\ \cdots}^{2m+1} \frac{\beta_k - d_k}{4} + \sum_{s=2,\ 4,\ \cdots}^{2m+2} \frac{\beta_s - d_s}{8} \geq f(H) \tag{1}$$

$$\alpha_1 + \frac{\alpha_2 - c_2}{2} \geq f(H) \tag{2}$$

$$\alpha_1 + \sum_{k=1,\ 3,\ \cdots}^{2m+1} \frac{\beta_k - d_k}{4} \geq f(H) \tag{3}$$

$$\alpha_1 + \sum_{s=2,\ 4,\ \cdots}^{2m+2} \frac{\beta_s - d_s}{8} \geq f(H) \tag{4}$$

$$\alpha_1 + \frac{\alpha_2 - c_2}{2} + \sum_{k=1,\ 3,\ \cdots}^{2m+1} \frac{\beta_k - d_k}{4} \geq f(H) \tag{5}$$

$$\alpha_1 + \frac{\alpha_2 - c_2}{2} + \sum_{s=2,\ 4,\ \cdots}^{2m+2} \frac{\beta_s - d_s}{8} \geq f(H) \tag{6}$$

$$\alpha_1 + \sum_{k=1,3}^{2m+1} \frac{\beta_k - d_k}{4} + \sum_{s=2,\ 4,\ \cdots}^{2m+2} \frac{\beta_s - d_s}{8} \geq f(H) \tag{7}$$

So we can transfer 1 pebble to $(u_1,\ x_1)$.

Case 2. Let $(u_2,\ x_k)$ or $(u_2,\ y_j)$ be the target vertex where $1 \le k \le 2$ and $1 \le j \le 2m$. Without loss of generality, let $(u_2,\ x_1)$ be the destination vertex. If $\alpha_2 \ge f(H)$, then we can transfer 1 pebble to the target. So let $\alpha_2 < f(H)$. If $\alpha_1 \ge 2f(H)$ or $\frac{\alpha_1 - c_1}{2} \ge f(H)$ and we can transfer 2 pebbles to $(u_1,\ x_1)$ then further we can move 1 pebble to the target. Otherwise if $\beta_k \ge 4f(H)$ where $k = \{1,\ 3,\ \cdots,\ 2m+1\}$ we can transfer 4 pebbles to $(v_k,\ x_1)$ and then 2 pebbles to $(u_1,\ x_1)$. Thus, we can reach the target. Otherwise we have $\beta_k \ge 4f(H)$ or $\frac{\beta_k - d_k}{8} \ge f(H)$. We can transfer 4 pebbles to $(v_k,\ x_1)$ and then 2 pebbles to $(u_1,\ x_1)$. Thus, we can reach the target. If $\beta_s \ge 2f(H)$ or $\frac{\beta_s - d_s}{4} \ge f(H)$ we can reach the destination. Suppose $\alpha_2 < f(H)$, $\frac{\alpha_2 - c_2}{2} < f(H)$, $\frac{\beta_k - d_k}{2} < 2f(H)$, $\frac{\beta_s - d_s}{2} < f(H)$ then we have one of the following:

$$\alpha_2 + \frac{\alpha_1 - c_1}{2} \ge f(H) \tag{8}$$

$$\alpha_2 + \sum_{k=1,\ 3,\ \cdots}^{2m+1} \frac{\beta_k - d_k}{8} \ge f(H) \tag{9}$$

$$\alpha_2 + \sum_{s=2,\ 4,\ \cdots}^{2m+2} \frac{\beta_s - d_s}{4} \ge f(H) \tag{10}$$

$$\alpha_2 + \frac{\alpha_1 - c_1}{2} + \sum_{k=1,\ 3,\ \cdots}^{2m+1} \frac{\beta_k - d_k}{8} \ge f(H) \tag{11}$$

$$\alpha_2 + \frac{\alpha_1 - c_1}{2} + \sum_{s=2,\ 4,\ \cdots}^{2m+2} \frac{\beta_s - d_s}{4} \ge f(H) \tag{12}$$

$$\alpha_2 + \sum_{k=1,3\cdots}^{2m+1} \frac{\beta_k - d_k}{8} + \sum_{s=2,\ 4,\ \cdots}^{2m+2} \frac{\beta_s - d_s}{8} \ge f(H) \tag{13}$$

$$\alpha_2 + \frac{\alpha_1 - c_1}{2} + \sum_{k=1,\ 3,\ \cdots}^{2m+1} \frac{\beta_k - d_k}{8} + \sum_{s=2,\ 4,\ \cdots}^{2m+2} \frac{\beta_s - d_s}{4} \ge f(H) \tag{14}$$

So we can transfer 1 pebble to $(u_2,\ x_1)$.

Case 3. Let $(v_k,\ x_s)$ or $(v_k,\ y_j)$ be the target vertex where $1 \leq s \leq 2$, $1 \leq k \leq 2n$ and $1 \leq j \leq 2m$.
Let $(v_k,\ x_1)$ be the destination vertex. Suppose $\beta_k \geq f(H)$, then we can transfer a pebbles to the target. So let $\beta_k < f(H)$. If $\beta_{k+1} \geq 2f(H)$ or $\frac{\beta_{k+1}-d_{k+1}}{2} \geq f(H)$ then we can transfer 2 pebbles to $(\beta_{k+1},\ x_1)$ and then one pebble to the target. Otherwise $\beta_{k+j} \geq 4f(H)$, where $k < j \leq 2n$ or $\frac{\beta_{k+j}-d_{k+j}}{8} \geq 2f(H)$. Then we can transfer 4 pebbles to $(v_{k+j},\ x_1)$ then 2 pebbles to $(v_{k+1},\ x_1)$. Thus, we can reach the target. If $\alpha_1 \geq 2f(H)$ or $\alpha_2 \geq 4f(H)$ and $(u_1,\ x_1)$ and $(u_1,\ y_j)$ are adjacent to the target we are done. Otherwise $\alpha_1 \geq 4f(H)$ or $\alpha_2 \geq 2f(H)$ and $(u_2,\ x_1)$ and $(u_2,\ y_j)$ are adjacent to the target and we are done. Suppose $\beta_k < f(H)$, $\beta_{k+1} < 2f(H)$, $\beta_{k+j} < 4f(H)$, $\alpha_2 < 4f(H)$ where $(u_1,\ x_1)$ and $(u_1,\ y_j)$ are adjacent to the target then the following equations are true:

$$\beta_k + \frac{\alpha_1 - c_1}{2} \geq f(H) \tag{15}$$

$$\beta_k + \frac{\alpha_2 - c_2}{4} \geq f(H) \tag{16}$$

$$\beta_k + \frac{\beta_{k+1} - d_{k+1}}{4} \geq f(H) \tag{17}$$

$$\beta_k + \sum_{k \neq j \text{ and } j \neq 1,\ j=2}^{2n} \frac{\beta_{k+j} - d_{k+j}}{8} \geq f(H) \tag{18}$$

$$\beta_k + \frac{\alpha_1 - c_1}{2} + \sum_{k \neq j \text{ and } j \neq 1,\ j=2}^{2n} \frac{\beta_{k+j} - d_{k+j}}{8} \geq f(H) \tag{19}$$

$$\beta_k + \frac{\alpha_2 - c_2}{4} + \sum_{k \neq j \text{ and } j \neq 1,\ j=2}^{2n} \frac{\beta_{k+j} - d_{k+j}}{8} \geq f(H) \tag{20}$$

$$\beta_k + \frac{\alpha_1 - c_1}{2} + \frac{\alpha_2 - c_2}{4} + \frac{\beta_{k+1} - d_{k+1}}{4} \geq f(H) \tag{21}$$

$$\beta_k + \frac{\alpha_1 - c_1}{2} + \frac{\alpha_2 - c_2}{4} + \sum_{k\neq j \ and \ j\neq 1,\ j=2}^{2n} \frac{\beta_{k+j} - d_{k+j}}{8} \geq f(H) \tag{22}$$

$$\beta_k + \frac{\alpha_1 - c_1}{2} + \frac{\beta_{k+1} - d_{k+1}}{4} + \sum_{k\neq j \ and \ j\neq 1,\ j=2}^{2n} \frac{\beta_{k+j} - d_{k+j}}{8} \geq f(H) \tag{23}$$

$$\beta_k + \frac{\alpha_2 - c_2}{4} + \frac{\beta_{k+1} - d_{k+1}}{4} + \sum_{k\neq j \ and \ j\neq 1,\ j=2}^{2n} \frac{\beta_{k+j} - d_{k+j}}{8} \geq f(H) \tag{24}$$

$$\beta_k + \frac{\alpha_1 - c_1}{2} + \frac{\alpha_2 - c_2}{4} + \frac{\beta_{k+1} - d_{k+1}}{4} + \sum_{k\neq j \ and \ j\neq 1,\ j=2}^{2n} \frac{\beta_{k+j} - d_{k+j}}{8} \geq f(H) \tag{25}$$

Then we can reach the target. ■

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