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Sum Divisor Cordial Labeling of Octopus, Friendship, and Graphs Derived from Fan and Complete Bipartite Graphs

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Abstract

A sum divisor cordial labeling of a graph G with vertex set $V(G)$ and edge set $E(G)$ is a bijection $f : V(G) \to \{1, 2, \ldots, |V(G)|\}$ such that each edge $uv \in E(G)$ receives the label 1 if $f(u) + f(v)$ is even, and 0 otherwise. A graph is called a sum divisor cordial graph if the numbers of edges labeled 0 and 1 differ by at most one.

In this work, we show that the graphs $\langle F_{1,m} \Delta F_{1,n} \rangle$ and the subdivision of the complete bipartite graph $S(K_{m,n})$ are sum divisor cordial for all natural numbers m and n. Furthermore, we investigate the sum divisor cordial labelings of the octopus graph O_n and the friendship graph F_n. We prove that O_n is a sum divisor cordial graph for all natural numbers n, whereas F_n admits such a labeling only for specific values of n.

Key words: Sum divisor cordial labeling, Sum divisor cordial graph, Subdivision, Complete bipartite graph, Octopus graph, Friendship graph

2010 Mathematics Subject Classification : 05C78

1 Introduction and Preliminaries

Graph labeling problems have been extensively studied in the literature and form an important branch of graph theory. Gallian [8] provides a well-known dynamic survey summarizing numerous labeling techniques and their developments. One such labeling scheme is the sum divisor cordial labeling introduced by Lourdusamy and Patrick [9]. This labeling has been investigated for several classes of graphs and graph operations; see, for instance, [1, 2, 3, 4, 5].

The octopus graph O_n and the friendship graph F_n are well-known graph families that have been extensively studied in various contexts of graph labeling. For instance, Ponraj and Gayathri [11] investigated the pair difference cordial labeling of the octopus graph O_n and proved that O_n admits such a labeling for $2 \leq n \leq 4$. The friendship graph F_n has also received considerable attention; in particular, Cahit [6] showed that F_n admits a cordial labeling except when $n \equiv 2 \pmod 4$.

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Several studies have focused on the sum divisor cordial labeling of graphs obtained through operations involving $F_{m,n}$ and $K_{m,n}$. For example, Sugumaran and Rajesh [15] examined the sum divisor cordial labeling of the umbrella graph $U(n, n)$ for odd integers $n > 2$, which is obtained by appending a path P_n to the central vertex of the fan graph $F_{1,n}$. Additionally, Lourdusamy and Patrick [9] studied the sum divisor cordial labeling of the subdivision of the star graph $S(K_{1,n})$ for all natural numbers n. Furthermore, Vaidya and Barasara [17] introduced a graph operation denoted by Δ, which has been used to investigate sum divisor cordial labelings of several graphs. For instance, Lourdusamy and Patrick [10] showed that the graph $\langle K_{1,n}\Delta K_{1,n}\rangle$ is sum divisor cordial for all natural numbers n. Subsequently, Sugumaran and Rajesh [14, 16] investigated the sum divisor cordial labeling of the graphs $\langle F_{1,n}\Delta F_{1,n}\rangle$, $\langle K_{1,n,n}\Delta K_{1,n,n}\rangle$, and $\langle Fl_n\Delta Fl_n\rangle$ for all natural numbers n.

In the present study, we extend the result of Lourdusamy and Patrick [9] by generalizing the sum divisor cordial labeling from the subdivision of the star graph $S(K_{1,n})$ to the subdivision of the complete bipartite graph $S(K_{m,n})$ for all natural numbers m and n. We also extend the result of Sugumaran and Rajesh [14] on the graph $\langle F_{1,n}\Delta F_{1,n}\rangle$ to the more general graph $\langle F_{1,m}\Delta F_{1,n}\rangle$ for all natural numbers m and n, where m and n are not necessarily equal, by modifying the definition of the graph operation Δ. Additionally, we show that the octopus graph O_n is sum divisor cordial for all natural numbers n, whereas the friendship graph F_n is sum divisor cordial only for specific values of n.

We begin by recalling some standard definitions, notation, and known results that will be used throughout this paper. All graphs considered in this work are finite, simple, nontrivial, and connected. For a graph G with vertex set $V(G)$ and edge set $E(G)$, the symbols $|V(G)|$ and $|E(G)|$ denote the numbers of vertices and edges of G, respectively. The symbol $\mathbb{N}$ denotes the set of all natural numbers. Unless otherwise stated, other graph-theoretic and number-theoretic terminology and notation not defined here follow those in [7, 12].

Definition 1.1. ([9]) Let G be a graph with vertex set $V(G)$ and edge set $E(G)$, and let $f : V(G) \to \{1, 2, \ldots, |V(G)|\}$ be a bijection. For each edge $uv \in E(G)$, define the induced function $f^* : E(G) \to \{0, 1\}$ by

$$f^*(uv) = \begin{cases} 1, & \text{if } f(u) + f(v) \text{ is even,} \\ 0, & \text{otherwise.} \end{cases}$$

The bijection f is called a *sum divisor cordial labeling* of G if $|e_f(0) - e_f(1)| \leq 1$, where $e_f(i)$ denotes the number of edges assigned label i. A graph which admits a sum divisor cordial labeling is called a *sum divisor cordial graph*.

Definition 1.2. ([11]) The *octopus graph* O_n is the graph with vertex set $V(O_n) = \{u, u_i, v_i : 1 \leq i \leq n\}$ and edge set $E(O_n) = \{u_iu_{i+1} : 1 \leq i \leq n-1\} \cup \{uu_i, uv_i : 1 \leq i \leq n\}$.

Definition 1.3. ([6]) A *friendship graph* F_n is a graph constructed by joining n copies of the cycle graph C_3 with a common vertex, which is called the apex vertex of the graph.

Definition 1.4. ([13]) A *fan graph* $F_{m,n}$ is the graph join $\overline{K}_m + P_n$, where $\overline{K}_m$ denotes the complement of K_m. For $m = 1$, this construction yields the *usual fan graph.*

Definition 1.5. ([10]) The *subdivision graph* $S(G)$ is obtained from G by subdividing each edge of G with a vertex.

Definition 1.6. Let G_1 and G_2 be two graphs, each containing an apex vertex, that is, a vertex adjacent to every other vertex of the graph (equivalently, a vertex of degree $|V(G_i)| - 1$). The graph $G' = \langle G_1\Delta G_2\rangle$ is constructed by joining the apex vertices of G_1 and G_2 with an edge and by connecting both apex vertices to a new vertex v.

Lemma 1.7. *([14]) The graph $\langle F_{1,n}\Delta F_{1,n}\rangle$ is a sum divisor cordial graph for all $n \in \mathbb{N}$.*

2 Main Results

Theorem 2.1. *The octopus graph O_n is a sum divisor cordial graph for all $n \in \mathbb{N}$.*

Proof: Let O_n be the octopus graph as defined in Definition 1.2. Then $|V(O_n)| = 2n+1$ and $|E(O_n)| = 3n - 1$.

For the small cases $n = 1, 2, 3$, we define bijections $f : V(O_n) \to \{1, 2, \ldots, 2n, 2n+1\}$ explicitly as follows.

For $n = 1$, define $f(u_1) = 1$, $f(u) = 3$, and $f(v_1) = 2$.

For $n = 2$, define $f(u_1) = 2$, $f(u_2) = 4$, $f(u) = 1$, and $f(v_1) = 3$, $f(v_2) = 5$.

For $n = 3$, define $f(u_1) = 2$, $f(u_2) = 4$, $f(u_3) = 5$, $f(u) = 1$, and $f(v_1) = 3$, $f(v_2) = 6$, $f(v_3) = 7$.

For the case $n \geq 4$, we write $n = 4t + r$ with $t \in \mathbb{N}$ and $0 \leq r \leq 3$, and define a bijection $f : V(O_n) \to \{1, 2, \ldots, 2n, 2n+1\}$ by setting $f(u) = 2n+1$, while the labels for the remaining vertices u_i and v_i for $1 \leq i \leq n$ are assigned according to the residue r as

specified in the following cases. **Case 1.** $r = 0$.

$$f(u_i) = \begin{cases} 4k-3, & i = 4k-3,\ 1 \le k \le \frac{n}{4}, \\ 4k-1, & i = 4k-2,\ 1 \le k \le \frac{n}{4}, \\ 4k-2, & i = 4k-1,\ 1 \le k \le \frac{n}{4}, \\ 4k, & i = 4k,\ 1 \le k \le \frac{n}{4}, \end{cases}$$

$$f(v_j) = \begin{cases} n+4k-3, & j = 4k-3,\ 1 \le k \le \frac{n}{4}, \\ n+4k-1, & j = 4k-2,\ 1 \le k \le \frac{n}{4}, \\ n+4k-2, & j = 4k-1,\ 1 \le k \le \frac{n}{4}, \\ n+4k, & j = 4k,\ 1 \le k \le \frac{n}{4}. \end{cases}$$

It follows that $e_f(1) = \frac{3n}{2}$ and $e_f(0) = \frac{3n}{2} - 1$. **Case 2.** $r = 1$.

$$f(u_i) = \begin{cases} 4k-3, & \text{if } i = 4k-3,\ 1 \le k \le \frac{n-1}{4}, \\ 4k-1, & \text{if } i = 4k-2,\ 1 \le k \le \frac{n-1}{4}, \\ 4k-2, & \text{if } i = 4k-1,\ 1 \le k \le \frac{n-1}{4}, \\ 4k, & \text{if } i = 4k,\ \ 1 \le k \le \frac{n-1}{4}, \\ n, & \text{if } i = n, \end{cases}$$

$$f(v_j) = \begin{cases} n+4k-3, & \text{if } j = 4k-3,\ 1 \le k \le \frac{n-1}{4}, \\ n+4k-1, & \text{if } j = 4k-2,\ 1 \le k \le \frac{n-1}{4}, \\ n+4k-2, & \text{if } j = 4k-1,\ 1 \le k \le \frac{n-1}{4}, \\ n+4k, & \text{if } j = 4k,\ \ 1 \le k \le \frac{n-1}{4}, \\ 2n, & \text{if } j = n. \end{cases}$$

Consequently, we obtain $e_f(1) = e_f(0) = \frac{3(n-1)}{2} + 1$. **Case 3.** $r = 2$.

$$f(u_i) = \begin{cases} 4k-3, & \text{if } i = 4k-3,\ 1 \le k \le \frac{n-2}{4}, \\ 4k-1, & \text{if } i = 4k-2,\ 1 \le k \le \frac{n-2}{4}, \\ 4k-2, & \text{if } i = 4k-1,\ 1 \le k \le \frac{n-2}{4}, \\ 4k, & \text{if } i = 4k,\ 1 \le k \le \frac{n-2}{4}, \\ n-1, & \text{if } i = n-1, \\ 2n-1, & \text{if } i = n, \end{cases}$$

$$f(v_j) = \begin{cases} n+4k-3, & \text{if } j = 4k-3,\ 1 \le k \le \frac{n-2}{4}, \\ n+4k-1, & \text{if } j = 4k-2,\ 1 \le k \le \frac{n-2}{4}, \\ n+4k-2, & \text{if } j = 4k-1,\ 1 \le k \le \frac{n-2}{4}, \\ n+4k, & \text{if } j = 4k,\ 1 \le k \le \frac{n-2}{4}, \\ n, & \text{if } j = n-1, \\ 2n, & \text{if } j = n. \end{cases}$$

Therefore, $e_f(1) = \frac{3(n-2)}{2} + 3$ and $e_f(0) = \frac{3(n-2)}{2} + 2$. **Case 4.** $r = 3$.

$$f(u_i) = \begin{cases} 4k-3, & \text{if } i = 4k-3,\ 1 \le k \le \frac{n-3}{4}, \\ 4k-1, & \text{if } i = 4k-2,\ 1 \le k \le \frac{n-3}{4}, \\ 4k-2, & \text{if } i = 4k-1,\ 1 \le k \le \frac{n-3}{4}, \\ 4k, & \text{if } i = 4k,\ 1 \le k \le \frac{n-3}{4}, \\ n-1, & \text{if } i = n-2, \\ n-2, & \text{if } i = n-1, \\ 2n, & \text{if } i = n, \end{cases}$$

$$f(v_j) = \begin{cases} n+4k-3, & \text{if } j = 4k-3,\ 1 \le k \le \frac{n-3}{4}, \\ n+4k-1, & \text{if } j = 4k-2,\ 1 \le k \le \frac{n-3}{4}, \\ n+4k-2, & \text{if } j = 4k-1,\ 1 \le k \le \frac{n-3}{4}, \\ n+4k, & \text{if } j = 4k,\ 1 \le k \le \frac{n-3}{4}, \\ n, & \text{if } j = n-2, \\ 2n-2, & \text{if } j = n-1, \\ 2n-1, & \text{if } j = n. \end{cases}$$

It is straightforward to verify that $e_f(1) = e_f(0) = \frac{3(n-3)}{2} + 4$.
In all cases considered above, we have $|e_f(0) - e_f(1)| \le 1$. ∎

Example 2.2. Figure 1 shows a sum divisor cordial labeling of O_5, where solid edges denote edges labeled 1 and dashed edges denote those labeled 0.

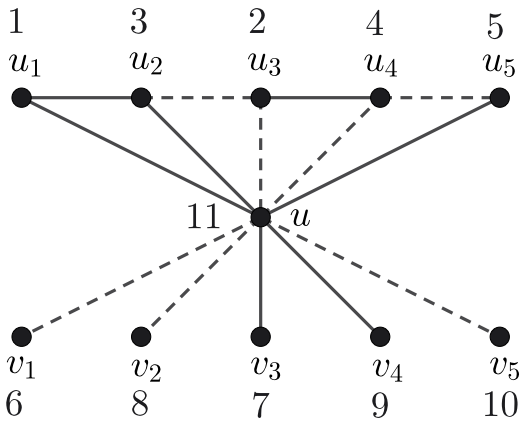

Figure 1: A sum divisor cordial labeling of O_5

Theorem 2.3. *The friendship graph F_n is a sum divisor cordial graph for all $n \in \mathbb{N}$, except when $n \equiv 2 \pmod 4$.*

Proof: Let $V(C_3^{(i)}) = \{x, x_1^{(i)}, x_2^{(i)}\}$ for $1 \leq i \leq n$, where the vertex x is the common vertex identified across all copies of $C_3^{(i)}$. It follows that

$$V(F_n) = \bigcup_{i=1}^{n} V(C_3^{(i)}), \quad |V(F_n)| = 2n+1, \quad \text{and} \quad |E(F_n)| = 3n.$$

Write $n = 4t + r$, where $t \in \mathbb{N} \cup \{0\}$ and $r \in \{0, 1, 3\}$. We define a bijection $f : V(F_n) \to \{1, 2, \ldots, 2n, 2n+1\}$ by setting $f(x) = 1$ and assigning the labels for the remaining vertices according to the following cases. **Case 1.** $r = 0$. For $i = 1, \ldots, \frac{n}{2}$, define

$$f\left(x_1^{(i)}\right) = 2i + 1, \qquad f\left(x_2^{(i)}\right) = 2i.$$

For $i = \frac{n}{2} + 1, \frac{n}{2} + 2, \ldots, \frac{3n}{4}$, define

$$f\left(x_1^{(i)}\right) = 4i - n - 1, \qquad f\left(x_2^{(i)}\right) = 4i - n + 1.$$

For $i = \frac{3n}{4} + 1, \frac{3n}{4} + 2, \ldots, n$, define

$$f\left(x_1^{(i)}\right) = 4i - 2n - 2, \qquad f\left(x_2^{(i)}\right) = 4i - 2n.$$

In this case, it can be shown that $e_f(1) = e_f(0) = \frac{3n}{2}$. **Case 2.** $r = 1$. For $i = 1, \ldots, \frac{n-1}{2} + 1$, define

$$f\left(x_1^{(i)}\right) = 2i + 1, \qquad f\left(x_2^{(i)}\right) = 2i.$$

For $i = \frac{n-1}{2} + 2, \frac{n-1}{2} + 3, \ldots, \frac{3(n-1)}{4} + 1$, define

$$f\left(x_1^{(i)}\right) = 4i - n - 3, \qquad f\left(x_2^{(i)}\right) = 4i - n - 1.$$

For $i = \frac{3(n-1)}{4} + 2, \frac{3(n-1)}{4} + 3, \ldots, n$, define

$$f\left(x_1^{(i)}\right) = 4i - 2n - 1, \qquad f\left(x_2^{(i)}\right) = 4i - 2n + 1.$$

It follows immediately that $e_f(1) = \frac{3n-1}{2}$ and $e_f(0) = \frac{3n+1}{2}$. **Case 3.** $r = 3$. For $i = 1, 2, \ldots, \frac{n-3}{2} + 1$, define

$$f\left(x_1^{(i)}\right) = 2i + 1, \qquad f\left(x_2^{(i)}\right) = 2i.$$

For $i = \frac{n-3}{2} + 2, \frac{n-3}{2} + 3$, define

$$f\left(x_1^{\left(\frac{n-3}{2}+2\right)}\right) = n + 3, \qquad f\left(x_2^{\left(\frac{n-3}{2}+2\right)}\right) = n + 1,$$

$$f\left(x_1^{\left(\frac{n-3}{2}+3\right)}\right) = n+4, \qquad f\left(x_2^{\left(\frac{n-3}{2}+3\right)}\right) = n+2.$$

For $i = \frac{n-3}{2}+4, \frac{n-3}{2}+5, \ldots, \frac{3(n-3)}{4}+3$, define

$$f\left(x_1^{(i)}\right) = 4i-n-4, \qquad f\left(x_2^{(i)}\right) = 4i-n-2.$$

For $i = \frac{3(n-3)}{4}+4, \frac{3(n-3)}{4}+5, \ldots, n$, define

$$f\left(x_1^{(i)}\right) = 4i-2n-2, \qquad f\left(x_2^{(i)}\right) = 4i-2n.$$

It is straightforward to verify that $e_f(1) = \frac{3n+1}{2}$ and $e_f(0) = \frac{3n-1}{2}$.
From the above cases, it follows that $|e_f(0) - e_f(1)| \leq 1$.

To establish that F_n is not a sum divisor cordial graph for $n = 4t+2$, where $t \in \mathbb{N} \cup \{0\}$, suppose to the contrary that there exists a sum divisor cordial labeling f. We proceed by considering the parity of $f(x)$.

If $f(x)$ is odd, the remaining available labels for the vertices $x_1^{(i)}$ and $x_2^{(i)}$ consist of exactly n odd and n even labels. This implies that among the $2n$ edges incident to the apex x, exactly n edges are labeled 0 and n edges are labeled 1. Let k_0 and k_1 denote the number of edges of the form $x_1^{(i)}x_2^{(i)}$ labeled 0 and 1, respectively. The cordial condition $|e_f(0) - e_f(1)| \leq 1$ then forces $k_0 = k_1 = n/2$. However, if $e_f(0) = n + n/2$, then

$$e_f(1) = n + \left\lfloor \frac{n/2}{2} \right\rfloor + \left\lfloor \frac{n/2}{2} \right\rfloor = n + \left(\frac{n}{2} - 1\right) < n + k_1,$$

which is a contradiction.

Similarly, let $f(x)$ be even. In this case, the remaining available labels for the vertices $x_1^{(i)}$ and $x_2^{(i)}$ consist of exactly $n+1$ odd labels and $n-1$ even labels. Consequently, among the edges incident to the apex, there are exactly $n+1$ edges labeled 0 and $n-1$ edges labeled 1. Let k_0 and k_1 be the number of edges of the form $x_1^{(i)}x_2^{(i)}$ receiving labels 0 and 1, respectively. To satisfy the condition $|e_f(0) - e_f(1)| \leq 1$, we must have $k_0 = n/2 - 1$ and $k_1 = n/2 + 1$. However, a direct calculation shows that

$$e_f(1) = (n-1) + \left\lfloor \frac{n/2}{2} \right\rfloor + \left\lfloor \frac{n/2+2}{2} \right\rfloor = (n-1) + \frac{n}{2},$$

which is strictly less than $(n-1) + k_1$, reaching a contradiction. ∎

Example 2.4. Figure 2 shows a sum divisor cordial labeling of F_4, where solid edges denote edges labeled 1 and dashed edges denote those labeled 0.

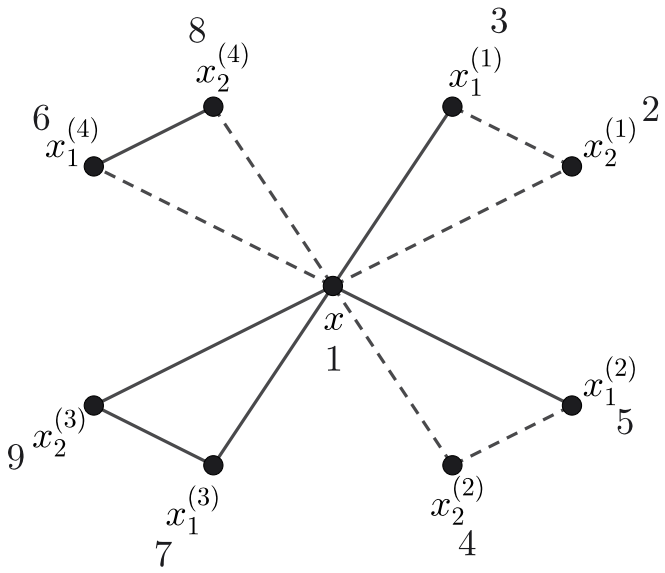

Figure 2: A sum divisor cordial labeling of F_4

Theorem 2.5. *The subdivision graph $S(K_{m,n})$ is a sum divisor cordial graph for all $m, n \in \mathbb{N}$.*

Proof: Let $V(S(K_{m,n})) = U \cup V \cup W$ where $U = \{u_1, u_2, \ldots, u_m\}$, $V = \{v_1, v_2, \ldots, v_n\}$, and $W = \{w_{i,j} \mid 1 \leq i \leq m, 1 \leq j \leq n\}$. Then $|V(S(K_{m,n}))| = m + n + mn$ and $|E(S(K_{m,n}))| = 2mn$. Define $f : V(S(K_{m,n})) \to \{1, 2, \ldots, m + n + mn - 1, m + n + mn\}$ according to the following cases. **Case 1.** $m = n$. For the vertices of U and V, we define

$$f(u_i) = 2i - 1, \quad 1 \leq i \leq m,$$
$$f(v_j) = 2j, \quad 1 \leq j \leq m.$$

For the subdivision vertices $w_{i,j} \in W$, define

$$f(w_{i,j}) = 2m + (i - 1)m + j, \quad 1 \leq i, j \leq m.$$

Case 2. $m > n$. For the vertices of U and V, we define

$$f(u_i) = 2i - 1, \quad 1 \leq i \leq m,$$
$$f(v_j) = 2j, \quad 1 \leq j \leq n.$$

For the subdivision vertices $w_{i,j} \in W$, let $k = (i-1)n+j$ for $1 \leq i \leq m$ and $1 \leq j \leq n$.

We define

$$f(w_{i,j}) = \begin{cases} 2n + 2k, & 1 \leq k \leq m - n, \\ 2m + (k - (m - n)), & m - n + 1 \leq k \leq mn. \end{cases}$$

Case 3. $m < n$. The case $m < n$ follows by symmetry from Case 2 by interchanging the roles of m and n and the vertex sets U and V.

In all cases, the labels assigned to the vertices of U are odd, while those assigned to the vertices of V are even. For any vertex $w_{i,j} \in W$, if $f(w_{i,j})$ is odd, then $f(u_i) + f(w_{i,j})$ is even and $f(v_j) + f(w_{i,j})$ is odd, which implies $f^*(u_i w_{i,j}) = 1$ and $f^*(w_{i,j} v_j) = 0$. If $f(w_{i,j})$ is even, then $f(u_i) + f(w_{i,j})$ is odd and $f(v_j) + f(w_{i,j})$ is even, yielding $f^*(u_i w_{i,j}) = 0$ and $f^*(w_{i,j} v_j) = 1$. Consequently, each vertex $w_{i,j}$ contributes exactly one edge with label 1 and one edge with label 0. Summing over all mn subdivision vertices, we obtain $e_f(1) = e_f(0) = mn$. Therefore, $|e_f(0) - e_f(1)| \leq 1$. ∎

Example 2.6. Figure 3 shows a sum divisor cordial labeling of $S(K_{3,2})$, where solid edges denote edges labeled 1 and dashed edges denote those labeled 0.

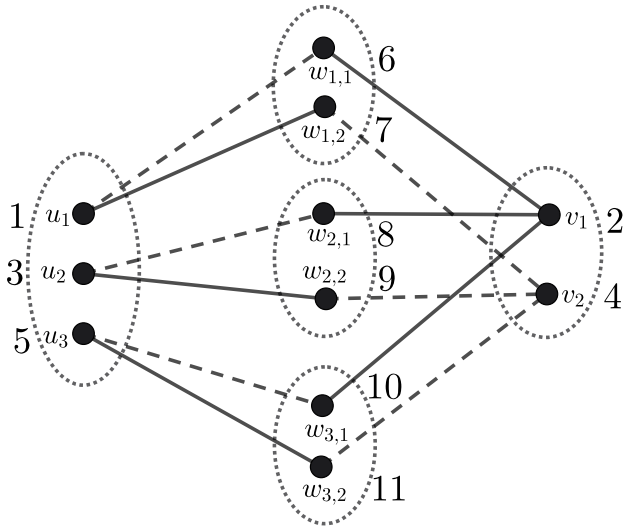

Figure 3: A sum divisor cordial labeling of $S(K_{3,2})$

Theorem 2.7. *The graph $\langle F_{1,m}\Delta F_{1,n}\rangle$ is a sum divisor cordial graph for all $m, n \in \mathbb{N}$.*

Proof: Let $V(F_{1,m}) = \{x, x_1, x_2, \ldots, x_m\}$ and $V(F_{1,n}) = \{y, y_1, y_2, \ldots, y_n\}$, where x and y are the apex vertices of $F_{1,m}$ and $F_{1,n}$, respectively, and let v be a new vertex. Then $|V(\langle F_{1,m}\Delta F_{1,n}\rangle)| = m + n + 3$ and $|E(\langle F_{1,m}\Delta F_{1,n}\rangle)| = 2m + 2n + 1$.

Note that the case $m = n$ has been established in Lemma 1.7. Hence, it remains to consider the case $m \neq n$. We divide the proof into two main parts. The first part deals with the cases $1 \leq m \leq 3$ or $1 \leq n \leq 3$, while the second part concerns $m, n \geq 4$.

We begin with the first part. Since the graphs $\langle F_{1,m}\Delta F_{1,n}\rangle$ and $\langle F_{1,n}\Delta F_{1,m}\rangle$ are isomorphic, without loss of generality, it suffices to consider the case $1 \leq m \leq 3$ with $m < n$. These cases are examined in detail through Cases 1 to 6 as follows. **Case 1.** $m = 1$ and $n = 2$. Define a bijection $f : V(\langle F_{1,1}\Delta F_{1,2}\rangle) \to \{1, 2, \ldots, 6\}$ by

$$f(x) = 1, \quad f(y) = 3, \quad f(v) = 5, \quad f(x_1) = 2, \quad f(y_1) = 4, \quad f(y_2) = 6.$$

Case 2. $m = 1$ and $n = 3$. Define a bijection $f : V(\langle F_{1,1}\Delta F_{1,3}\rangle) \to \{1, 2, \ldots, 7\}$ by

$$f(x) = 1, \quad f(y) = 3, \quad f(v) = 5, \quad f(x_1) = 2, \quad f(y_1) = 4, \quad f(y_2) = 6, \quad f(y_3) = 7.$$

Case 3. $m = 1$ and $n = 4t + r$, where $t \in \mathbb{N}$ and $0 \leq r \leq 3$. Define a bijection $f : V(\langle F_{1,1}\Delta F_{1,n}\rangle) \to \{1, 2, \ldots, n + 3, n + 4\}$ according to the value of r, as described in the following subcases. *Subcase 3.1.* $r = 0$. For the vertices x, y, v, and x_1, define $f(x) = n + 1, f(y) = n + 2, f(v) = n + 4$, and $f(x_1) = n + 3$.

For the vertices $y_1, y_2, \ldots, y_n$, define $f(y_j) = \begin{cases} 4k-3, & \text{if } j = 4k-3,\ 1 \leq k \leq \frac{n}{4}, \\ 4k-1, & \text{if } j = 4k-2,\ 1 \leq k \leq \frac{n}{4}, \\ 4k-2, & \text{if } j = 4k-1,\ 1 \leq k \leq \frac{n}{4}, \\ 4k, & \text{if } j = 4k,\ 1 \leq k \leq \frac{n}{4}. \end{cases}$ *Subcase 3.2.* $r = 1$. For the vertices x, y, v, and x_1, define $f(x) = n + 2, f(y) = n + 1, f(v) = n + 3$, and $f(x_1) = n + 4$.

For the vertices $y_1, y_2, \ldots, y_n$, define $f(y_j) = \begin{cases} 4k-3, & \text{if } j = 4k-3,\ 1 \leq k \leq \frac{n-1}{4}, \\ 4k-1, & \text{if } j = 4k-2,\ 1 \leq k \leq \frac{n-1}{4}, \\ 4k-2, & \text{if } j = 4k-1,\ 1 \leq k \leq \frac{n-1}{4}, \\ 4k, & \text{if } j = 4k,\ 1 \leq k \leq \frac{n-1}{4}, \\ n, & \text{if } j = n. \end{cases}$

Subcase 3.3. $r = 2$. For the vertices x, y, v, and x_1, define f as in Subcase 3.1.

For the vertices $y_1, y_2, \ldots, y_n$, define $f(y_j) = \begin{cases} 4k-3, & \text{if } j = 4k-3,\ 1 \leq k \leq \frac{n-2}{4}, \\ 4k-1, & \text{if } j = 4k-2,\ 1 \leq k \leq \frac{n-2}{4}, \\ 4k-2, & \text{if } j = 4k-1,\ 1 \leq k \leq \frac{n-2}{4}, \\ 4k, & \text{if } j = 4k,\ 1 \leq k \leq \frac{n-2}{4}, \\ n-1, & \text{if } j = n-1, \\ n, & \text{if } j = n. \end{cases}$

Subcase 3.4. $r = 3$. For the vertices x, y, v, and x_1, define f as in Subcase 3.2.

For the vertices $y_1, y_2, \ldots, y_n$, define $f(y_j) = \begin{cases} 4k-3, & \text{if } j = 4k-3,\ 1 \le k \le \frac{n-3}{4}, \\ 4k-1, & \text{if } j = 4k-2,\ 1 \le k \le \frac{n-3}{4}, \\ 4k-2, & \text{if } j = 4k-1,\ 1 \le k \le \frac{n-3}{4}, \\ 4k, & \text{if } j = 4k,\ 1 \le k \le \frac{n-3}{4}, \\ n-1, & \text{if } j = n-2, \\ n-2, & \text{if } j = n-1, \\ n, & \text{if } j = n. \end{cases}$

A straightforward verification shows that $e_f(1) = n+2$ and $e_f(0) = n+1$ in Subcases 3.1 and 3.4, whereas $e_f(1) = n+1$ and $e_f(0) = n+2$ in the remaining subcases. **Case 4.** $m = 2$ and $n = 3$. Define a bijection $f : V(\langle F_{1,2}\Delta F_{1,3}\rangle) \to \{1, 2, \ldots, 8\}$ by

$$f(x) = 1, \quad f(y) = 3, \quad f(v) = 5, \quad f(x_1) = 2, \quad f(x_2) = 4, \quad f(y_1) = 6, \quad f(y_2) = 8, \quad f(y_3) = 7.$$

Case 5. $m = 2$ and $n = 4t + r$, where $t \in \mathbb{N}$ and $0 \le r \le 3$. Define a bijection $f : V(\langle F_{1,2}\Delta F_{1,n}\rangle) \to \{1, 2, \ldots, n+4, n+5\}$ according to the value of r, as described in the following subcases. *Subcase 5.1.* $r = 0$. For the vertices x, y, v, and x_1, x_2, define $f(x) = n+2, f(y) = n+4, f(v) = n+1$, and $f(x_1) = n+3, f(x_2) = n+5$.

For the vertices $y_1, y_2, \ldots, y_n$, define f as in Subcase 3.1. *Subcase 5.2.* $r = 1$. For the vertices x, y, v, and x_1, x_2, define f as in Subcase 5.1.

For the vertices $y_1, y_2, \ldots, y_n$, define f as in Subcase 3.2. *Subcase 5.3.* $r = 2$. For the vertices x, y, v, and x_1, x_2, define $f(x) = n, f(y) = n+2, f(v) = n+4$, and $f(x_1) = n+3, f(x_2) = n+5$.

For the vertices $y_1, y_2, \ldots, y_n$, define $f(y_j) = \begin{cases} 4k-3, & \text{if } j = 4k-3,\ 1 \le k \le \frac{n-2}{4}, \\ 4k-1, & \text{if } j = 4k-2,\ 1 \le k \le \frac{n-2}{4}, \\ 4k-2, & \text{if } j = 4k-1,\ 1 \le k \le \frac{n-2}{4}, \\ 4k, & \text{if } j = 4k,\ 1 \le k \le \frac{n-2}{4}, \\ n-1, & \text{if } j = n-1, \\ n+1, & \text{if } j = n. \end{cases}$

Subcase 5.4. $r = 3$. For the vertices x, y, v, and x_1, x_2, define $f(x) = n, f(y) = n+1, f(v) = n+4$, and $f(x_1) = n+3, f(x_2) = n+5$.

For the vertices $y_1, y_2, \ldots, y_n$, define $f(y_j) = \begin{cases} 4k-3, & \text{if } j = 4k-3,\ 1 \le k \le \frac{n-3}{4}, \\ 4k-1, & \text{if } j = 4k-2,\ 1 \le k \le \frac{n-3}{4}, \\ 4k-2, & \text{if } j = 4k-1,\ 1 \le k \le \frac{n-3}{4}, \\ 4k, & \text{if } j = 4k,\ 1 \le k \le \frac{n-3}{4}, \\ n-1, & \text{if } j = n-2, \\ n-2, & \text{if } j = n-1, \\ n+2, & \text{if } j = n. \end{cases}$

By direct computation, we obtain that $e_f(1) = n+3$ and $e_f(0) = n+2$ in Subcase 5.3, while $e_f(1) = n+2$ and $e_f(0) = n+3$ in all remaining subcases. **Case 6.** $m = 3$ and $n = 4t+r$,

where $t \in \mathbb{N}$ and $0 \leq r \leq 3$. Define a bijection $f : V(\langle F_{1,3}\Delta F_{1,n}\rangle) \to \{1, 2, \ldots, n+5, n+6\}$ according to the value of r, as described in the following subcases. *Subcase 6.1.* $r = 0$. For the vertices x, y, v, and x_1, x_2, x_3, define $f(x) = n+1, f(y) = n+2, f(v) = n+4$, and $f(x_1) = n+3, f(x_2) = n+5, f(x_3) = n+6$.

For the vertices $y_1, y_2, \ldots, y_n$, define f as in Subcase 3.1. *Subcase 6.2.* $r = 1$. For the vertices x, y, v, and x_1, x_2, x_3, define $f(x) = n+2, f(y) = n+1, f(v) = n+3$, and $f(x_1) = n+4, f(x_2) = n+6, f(x_3) = n+5$.

For the vertices $y_1, y_2, \ldots, y_n$, define f as in Subcase 3.2. *Subcase 6.3.* $r = 2$. For the vertices x, y, v, and x_1, x_2, x_3, define f as in Subcase 6.1.

For the vertices $y_1, y_2, \ldots, y_n$, define f as in Subcase 3.3. *Subcase 6.4.* $r = 3$. For the vertices x, y, v, and x_1, x_2, x_3, define $f(x) = n+1, f(y) = n+2, f(v) = n+3$, and $f(x_1) = n+4, f(x_2) = n+6, f(x_3) = n+5$.

For the vertices $y_1, y_2, \ldots, y_n$, define $f(y_j) = \begin{cases} 4k-3, & \text{if } j = 4k-3,\ 1 \leq k \leq \frac{n-3}{4}, \\ 4k-1, & \text{if } j = 4k-2,\ 1 \leq k \leq \frac{n-3}{4}, \\ 4k-2, & \text{if } j = 4k-1,\ 1 \leq k \leq \frac{n-3}{4}, \\ 4k, & \text{if } j = 4k,\ 1 \leq k \leq \frac{n-3}{4}, \\ n-2, & \text{if } j = n-2, \\ n, & \text{if } j = n-1, \\ n-1, & \text{if } j = n. \end{cases}$

A direct computation shows that $e_f(1) = n+4$ and $e_f(0) = n+3$ in Subcase 6.1, while $e_f(1) = n+3$ and $e_f(0) = n+4$ in all remaining subcases.

We now turn to the second part, where $m, n \geq 4$. Since the graphs $\langle F_{1,m}\Delta F_{1,n}\rangle$ and $\langle F_{1,n}\Delta F_{1,m}\rangle$ are isomorphic, we may assume without loss of generality that $m < n$. The labeling constructions follow the residue classes of m and n modulo 4. Under these assumptions, it suffices to consider the following cases, which are analyzed in detail in Cases 7 through 10. For each case, we define a bijection $f : V(\langle F_{1,m}\Delta F_{1,n}\rangle) \to \{1, 2, \ldots, m+n+2, m+n+3\}$ as follows. **Case 7.** Let $m = 4t_1$ and $n = 4t_2 + r$, where $t_1, t_2 \in \mathbb{N}$ and $0 \leq r \leq 3$. For the vertices $x_1, x_2, \ldots, x_m$, define $f(x_i) = \begin{cases} 4k-3, & \text{if } i = 4k-3,\ 1 \leq k \leq \frac{m}{4}, \\ 4k-1, & \text{if } i = 4k-2,\ 1 \leq k \leq \frac{m}{4}, \\ 4k-2, & \text{if } i = 4k-1,\ 1 \leq k \leq \frac{m}{4}, \\ 4k, & \text{if } i = 4k,\ 1 \leq k \leq \frac{m}{4}. \end{cases}$

For the vertices x, y, v and $y_1, y_2, \ldots, y_n$, the labeling f is defined in subcases depending on the value of r. *Subcase 7.1.* $r = 0$. Define $f(x) = m+n+1, f(y) = m+n+2, f(v) = m+n+3$, and

$$f(y_j) = \begin{cases} m+4k-3, & \text{if } j = 4k-3,\ 1 \le k \le \frac{n}{4}, \\ m+4k-1, & \text{if } j = 4k-2,\ 1 \le k \le \frac{n}{4}, \\ m+4k-2, & \text{if } j = 4k-1,\ 1 \le k \le \frac{n}{4}, \\ m+4k, & \text{if } j = 4k,\ 1 \le k \le \frac{n}{4}. \end{cases}$$

Subcase 7.2. $r = 1$. For the vertices x, y, v, the labels are defined as in Subcase 7.1, while for the vertices $y_1, y_2, \ldots, y_n$, the mapping f is given by

$$f(y_j) = \begin{cases} m+4k-3, & \text{if } j = 4k-3,\ 1 \le k \le \frac{n-1}{4}, \\ m+4k-1, & \text{if } j = 4k-2,\ 1 \le k \le \frac{n-1}{4}, \\ m+4k-2, & \text{if } j = 4k-1,\ 1 \le k \le \frac{n-1}{4}, \\ m+4k, & \text{if } j = 4k,\ 1 \le k \le \frac{n-1}{4}, \\ m+n, & \text{if } j = n. \end{cases}$$

Subcase 7.3. $r = 2$. Define $f(x) = m+n+1, f(y) = m+n+3, f(v) = m+n+2$, and

$$f(y_j) = \begin{cases} m+4k-3, & \text{if } j = 4k-3,\ 1 \le k \le \frac{n-2}{4}, \\ m+4k-1, & \text{if } j = 4k-2,\ 1 \le k \le \frac{n-2}{4}, \\ m+4k-2, & \text{if } j = 4k-1,\ 1 \le k \le \frac{n-2}{4}, \\ m+4k, & \text{if } j = 4k,\ 1 \le k \le \frac{n-2}{4}, \\ m+n-1, & \text{if } j = n-1, \\ m+n, & \text{if } j = n. \end{cases}$$

Subcase 7.4. $r = 3$. Define $f(x) = m+n, f(y) = m+n+3, f(v) = m+n+2$, and

$$f(y_j) = \begin{cases} m+4k-3, & \text{if } j = 4k-3,\ 1 \le k \le \frac{n-3}{4}, \\ m+4k-1, & \text{if } j = 4k-2,\ 1 \le k \le \frac{n-3}{4}, \\ m+4k-2, & \text{if } j = 4k-1,\ 1 \le k \le \frac{n-3}{4}, \\ m+4k, & \text{if } j = 4k,\ 1 \le k \le \frac{n-3}{4}, \\ m+n-2, & \text{if } j = n-2, \\ m+n-1, & \text{if } j = n-1, \\ m+n+1, & \text{if } j = n. \end{cases}$$

By direct computation, we have $e_f(1) = m+n$ and $e_f(0) = m+n+1$ in Subcase 7.3, while $e_f(1) = m+n+1$ and $e_f(0) = m+n$ in all remaining subcases. **Case 8.** Let $m = 4t_1+1$ and $n = 4t_2+r$, where $t_1, t_2 \in \mathbb{N}$ and $1 \le r \le 3$. For the vertices $x_1, x_2, \ldots, x_m$, define

$$f(x_i) = \begin{cases} 4k-3, & \text{if } i = 4k-3,\ 1 \le k \le \frac{m-1}{4}, \\ 4k-1, & \text{if } i = 4k-2,\ 1 \le k \le \frac{m-1}{4}, \\ 4k-2, & \text{if } i = 4k-1,\ 1 \le k \le \frac{m-1}{4}, \\ 4k, & \text{if } i = 4k,\ 1 \le k \le \frac{m-1}{4}, \\ m, & \text{if } i = m. \end{cases}$$

The labeling of the remaining vertices x, y, v, and $y_1, \ldots, y_n$ depends on the value of r as specified in the following subcases. *Subcase 8.1.* $r = 1$. Define $f(x) = m + n + 3$, $f(y) = m + n$, $f(v) = m + n + 2$, and

$$f(y_j) = \begin{cases} m + 4k - 2, & \text{if } j = 4k - 3,\ 1 \le k \le \frac{n-1}{4}, \\ m + 4k, & \text{if } j = 4k - 2,\ 1 \le k \le \frac{n-1}{4}, \\ m + 4k - 3, & \text{if } j = 4k - 1,\ 1 \le k \le \frac{n-1}{4}, \\ m + 4k - 1, & \text{if } j = 4k,\ 1 \le k \le \frac{n-1}{4}, \\ m + n + 1, & \text{if } j = n. \end{cases}$$

Subcase 8.2. $r = 2$. Define $f(x) = m + n + 2$, $f(y) = m + n + 1$, $f(v) = m + n + 3$, and

$$f(y_j) = \begin{cases} m + 4k - 2, & \text{if } j = 4k - 3,\ 1 \le k \le \frac{n-2}{4}, \\ m + 4k, & \text{if } j = 4k - 2,\ 1 \le k \le \frac{n-2}{4}, \\ m + 4k - 3, & \text{if } j = 4k - 1,\ 1 \le k \le \frac{n-2}{4}, \\ m + 4k - 1, & \text{if } j = 4k,\ 1 \le k \le \frac{n-2}{4}, \\ m + n, & \text{if } j = n - 1, \\ m + n - 1, & \text{if } j = n. \end{cases}$$

Subcase 8.3. $r = 3$. For the vertices x, y, v, the labels are defined as in Subcase 8.1, while for the vertices $y_1, y_2, \ldots, y_n$, the mapping f is given by

$$f(y_j) = \begin{cases} m + 4k - 2, & \text{if } j = 4k - 3,\ 1 \le k \le \frac{n-3}{4}, \\ m + 4k, & \text{if } j = 4k - 2,\ 1 \le k \le \frac{n-3}{4}, \\ m + 4k - 3, & \text{if } j = 4k - 1,\ 1 \le k \le \frac{n-3}{4}, \\ m + 4k - 1, & \text{if } j = 4k,\ 1 \le k \le \frac{n-3}{4}, \\ m + n - 1, & \text{if } j = n - 2, \\ m + n + 1, & \text{if } j = n - 1, \\ m + n - 2, & \text{if } j = n. \end{cases}$$

A straightforward calculation shows that $e_f(1) = m + n$ and $e_f(0) = m + n + 1$ in all subcases. **Case 9.** Let $m = 4t_1 + 2$ and $n = 4t_2 + r$, where $t_1, t_2 \in \mathbb{N}$ and $2 \le r \le 3$. *Subcase 9.1.* $r = 2$. For the vertices x, y, and v, define f as in Subcase 8.1.

For the vertices $x_1, x_2, \ldots, x_m$, define $f(x_i) = \begin{cases} 4k - 3, & \text{if } i = 4k - 3,\ 1 \le k \le \frac{m-2}{4}, \\ 4k - 1, & \text{if } i = 4k - 2,\ 1 \le k \le \frac{m-2}{4}, \\ 4k - 2, & \text{if } i = 4k - 1,\ 1 \le k \le \frac{m-2}{4}, \\ 4k, & \text{if } i = 4k,\ 1 \le k \le \frac{m-2}{4}, \\ m - 1, & \text{if } i = m - 1, \\ m + n - 1, & \text{if } i = m. \end{cases}$

For the vertices $y_1, y_2, \ldots, y_n$, define $f(y_j) = \begin{cases} m+4k-3, & \text{if } j = 4k-3,\ 1 \le k \le \frac{n-2}{4}, \\ m+4k-1, & \text{if } j = 4k-2,\ 1 \le k \le \frac{n-2}{4}, \\ m+4k-2, & \text{if } j = 4k-1,\ 1 \le k \le \frac{n-2}{4}, \\ m+4k, & \text{if } j = 4k,\ 1 \le k \le \frac{n-2}{4}, \\ m+n+1, & \text{if } j = n-1, \\ m, & \text{if } j = n. \end{cases}$

Subcase 9.2. $r = 3$. For the vertices x, y, and v, define $f(x) = m+n-1$, $f(y) = m+n+1$, and $f(v) = m+n+3$.

For the vertices $x_1, x_2, \ldots, x_m$, define $f(x_i) = \begin{cases} 4k-3, & \text{if } i = 4k-3,\ 1 \le k \le \frac{m-2}{4}, \\ 4k-1, & \text{if } i = 4k-2,\ 1 \le k \le \frac{m-2}{4}, \\ 4k-2, & \text{if } i = 4k-1,\ 1 \le k \le \frac{m-2}{4}, \\ 4k, & \text{if } i = 4k,\ 1 \le k \le \frac{m-2}{4}, \\ m-1, & \text{if } i = m-1, \\ m+n, & \text{if } i = m. \end{cases}$

For the vertices $y_1, y_2, \ldots, y_n$, define $f(y_j) = \begin{cases} m+4k-3, & \text{if } j = 4k-3,\ 1 \le k \le \frac{n-3}{4}, \\ m+4k-1, & \text{if } j = 4k-2,\ 1 \le k \le \frac{n-3}{4}, \\ m+4k-2, & \text{if } j = 4k-1,\ 1 \le k \le \frac{n-3}{4}, \\ m+4k, & \text{if } j = 4k,\ 1 \le k \le \frac{n-3}{4}, \\ m+n-2, & \text{if } j = n-2, \\ m+n+2, & \text{if } j = n-1, \\ m, & \text{if } j = n. \end{cases}$

Consequently, we obtain $e_f(1) = m+n+1$ and $e_f(0) = m+n$ in all subcases. **Case 10.** Let $m = 4t_1+3$ and $n = 4t_2+3$, where $t_1, t_2 \in \mathbb{N}$. For the vertices x, y, and v, define f as in Subcase 8.1.

For the vertices $x_1, x_2, \ldots, x_m$, define $f(x_i) = \begin{cases} 4k-3, & \text{if } i = 4k-3,\ 1 \le k \le \frac{m-3}{4}, \\ 4k-1, & \text{if } i = 4k-2,\ 1 \le k \le \frac{m-3}{4}, \\ 4k-2, & \text{if } i = 4k-1,\ 1 \le k \le \frac{m-3}{4}, \\ 4k, & \text{if } i = 4k,\ 1 \le k \le \frac{m-3}{4}, \\ m-2, & \text{if } i = m-2, \\ m, & \text{if } i = m-1, \\ m-1, & \text{if } i = m. \end{cases}$

For the vertices $y_1, y_2, \ldots, y_n$, define f as in Subcase 8.3.

Therefore, $e_f(1) = m+n$ and $e_f(0) = m+n+1$.

From the above cases and their corresponding subcases, we conclude that $|e_f(0) - e_f(1)| \le 1$. ∎

Example 2.8. Figure 4 shows a sum divisor cordial labeling of $\langle F_{1,4}\Delta F_{1,5}\rangle$, where solid edges denote edges labeled 1 and dashed edges denote those labeled 0.

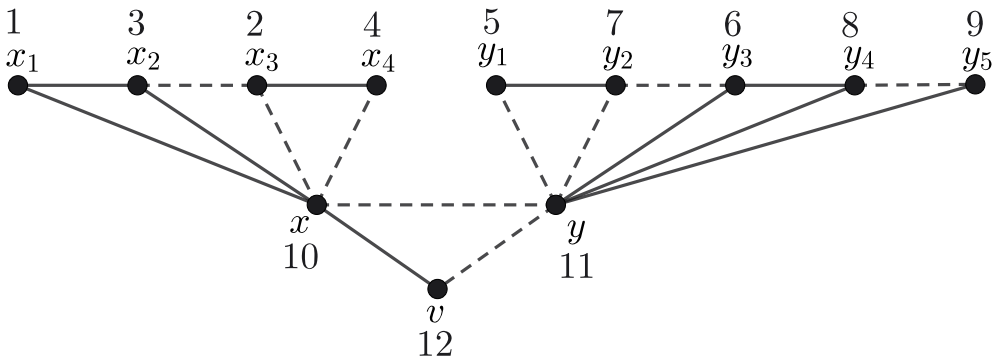

Figure 4: A sum divisor cordial labeling of $\langle F_{1,4} \Delta F_{1,5} \rangle$

3 Concluding Remarks

In this work, the sum divisor cordial labeling of the graphs $\langle F_{1,m} \Delta F_{1,n} \rangle$, the subdivision of the complete bipartite graph $S(K_{m,n})$, the friendship graph F_n, and the octopus graph O_n was investigated. Explicit constructions of such labelings were provided for each graph.

We showed that $\langle F_{1,m} \Delta F_{1,n} \rangle$ and $S(K_{m,n})$ are sum divisor cordial for all natural numbers m and n, and that the octopus graph O_n is sum divisor cordial for all natural numbers n. In contrast, the friendship graph F_n admits a sum divisor cordial labeling only for certain values of n. These results may serve as a basis for further investigations of sum divisor cordial labeling for other classes of graphs and related graph operations.

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