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On the Sum Divisor Cordial Labeling of Pagoda, Diamond Ladder, Jellyfish, and D_n^* Graphs

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Abstract

A sum divisor cordial labeling of a graph G with vertex set $V(G)$ and edge set $E(G)$ is a bijection $f : V(G) \to \{1, 2, \ldots, |V(G)|\}$ such that each edge $uv \in E(G)$ receives the label 1 if $f(u) + f(v)$ is even, and 0 otherwise. A graph is called a sum divisor cordial graph if the numbers of edges labeled 0 and 1 differ by at most one. In this paper, we prove that the pagoda graph PG_n, the diamond ladder graph DL_n, and the graph D_n^* are sum divisor cordial graphs for all positive integers n. Furthermore, we establish that the jellyfish graph $J(m, n)$ is a sum divisor cordial graph for all positive integers m and n.

Key words: Sum divisor cordial labeling, Sum divisor cordial graph, Pagoda graph, Diamond ladder graph, Jellyfish graph

2020 Mathematics Subject Classification: 05C78

1 Introduction and Preliminaries

Graph labeling has been widely investigated in the literature and remains an active area of research in graph theory. Among various labeling techniques, the concept of sum divisor cordial labeling was introduced by Lourdusamy and Patrick [10]. Subsequently, sum divisor cordial labeling has been extensively studied for various graph families as well as graph operations [1, 2, 3, 4, 5, 8, 11, 16, 17]. For a comprehensive survey of graph labeling problems, including sum divisor cordial labeling, we refer the reader to Gallian's dynamic survey [9].

The pagoda graph PG_n, the diamond ladder graph DL_n, the graph D_n^*, and the jellyfish graph $J(m, n)$ possess interesting structural properties and have been investigated under various graph labeling schemes. For instance, Ponraj and Gayathri [13] studied pair difference cordial labeling for these structures, proving that the pagoda graph PG_n is pair difference cordial for all $n \geq 3$, the diamond ladder graph DL_n is pair difference cordial for all $n \geq 2$, and the graph D_n^* is pair difference cordial for all $n \geq 3$. Additionally,

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the jellyfish graph $J(m,n)$ was investigated under pair sum labeling by Manimekalai and Thirusangu [12], who proved that $J(m,n)$ is a pair sum graph for all $m,n \geq 1$. It was also studied in the context of divisor cordial labeling by Seoud and Salman [15], who established that $J(m,n)$ is a divisor cordial graph for all $m,n \geq 1$.

Furthermore, Christy and Palani [7] studied the sum divisor cordial labeling of the jellyfish graph $J(m,n)$ for the case where $m = n$ with even positive integers m and n.

Motivated by these results, in this paper, we investigate sum divisor cordial labeling for all four graph families. Specifically, we construct explicit sum divisor cordial labelings for the pagoda graph PG_n, the diamond ladder graph DL_n, and the graph D_n^* for all positive integers n, as well as for the jellyfish graph $J(m,n)$ for all positive integers m and n, thereby extending the result of Christy and Palani [7] to the general case where m and n are not necessarily equal.

We now recall standard definitions and notation that will be used throughout the paper. All graphs considered in this work are finite, simple, nontrivial, and connected. For a graph G with vertex set $V(G)$ and edge set $E(G)$, $|V(G)|$ and $|E(G)|$ denote the number of vertices and edges of G, respectively. Here, $\mathbb{N}$ denotes the set of all positive integers. Unless otherwise specified, all graph-theoretic and number-theoretic terminology and notation follow those in [6, 14].

Definition 1.1. ([10]) Let G be a graph with vertex set $V(G)$ and edge set $E(G)$, and let $f : V(G) \to \{1, 2, \ldots, |V(G)|\}$ be a bijection. For each edge $uv \in E(G)$, the induced edge labeling $f^* : E(G) \to \{0, 1\}$ is defined by

$$f^*(uv) = \begin{cases} 1, & \text{if } f(u) + f(v) \text{ is even,} \\ 0, & \text{otherwise.} \end{cases}$$

The function f is called a *sum divisor cordial labeling* of G if $|e_f(0) - e_f(1)| \leq 1$, where $e_f(i)$ denotes the number of edges assigned label i under f^* for $i \in \{0, 1\}$. A graph that admits a sum divisor cordial labeling is called a *sum divisor cordial graph*.

Definition 1.2. ([13]) The *pagoda graph* PG_n, for $n \in \mathbb{N}$, is the graph with vertex set $V(PG_n) = \{x, x_i, y_i : 1 \leq i \leq n\}$ and edge set $E(PG_n) = \{x_i x_{i+1}, y_i y_{i+1} : 1 \leq i \leq n-1\} \cup \{x_i y_i : 1 \leq i \leq n\} \cup \{x_1 x, y_1 x\}$. Thus, $|V(PG_n)| = 2n + 1$ and $|E(PG_n)| = 3n$.

Definition 1.3. ([13]) The *diamond ladder graph* is the graph obtained from ladder L_n and it is denoted by DL_n. Let $V(DL_n) = \{x_i : 1 \leq i \leq n\} \cup \{y_i : 1 \leq i \leq n\} \cup \{z_i : 1 \leq i \leq 2n\}$ and $E(DL_n) = \{x_i x_{i+1}, y_i y_{i+1} : 1 \leq i \leq n-1\} \cup \{x_i y_i : 1 \leq i \leq n\} \cup \{x_i z_{2i-1}, y_i z_{2i-1} : 1 \leq i \leq n\} \cup \{x_i z_{2i}, y_i z_{2i} : 1 \leq i \leq n\} \cup \{z_{2i} z_{2i+1} : 1 \leq i \leq n-1\}$. It is easy to verify that $|V(DL_n)| = 4n$ and $|E(DL_n)| = 8n - 3$.

Definition 1.4. ([13]) The graph D_n^* is the graph with the vertex set $V(D_n^*) = \{x_i : 1 \leq i \leq n\} \cup \{y_i : 1 \leq i \leq n\} \cup \{z_i : 1 \leq i \leq 2n\}$ and the edge set $E(D_n^*) = \{x_ix_{i+1}, y_iy_{i+1} : 1 \leq i \leq n-1\} \cup \{x_iz_{2i-1}, y_iz_{2i-1} : 1 \leq i \leq n\} \cup \{x_iz_{2i}, y_iz_{2i} : 1 \leq i \leq n\} \cup \{z_{2i}z_{2i+1} : 1 \leq i \leq n-1\}$. For this graph, $|V(D_n^*)| = 4n$ and $|E(D_n^*)| = 7n-3$.

Definition 1.5. ([15]) The *jellyfish graph* $J(m,n)$, where $m,n \in \mathbb{N}$, is obtained from a 4-cycle $v_1v_2v_3v_4v_1$ by adding the edge v_1v_3, and attaching m pendant vertices to v_2 and n pendant vertices to v_4. Observe that $|V(J(m,n))| = m+n+4$ and $|E(J(m,n))| = m+n+5$.

2 Main Results

Theorem 2.1. *The pagoda graph PG_n is a sum divisor cordial graph for all $n \in \mathbb{N}$.*

Proof: By Definition 1.2, the pagoda graph PG_n has $|V(PG_n)| = 2n+1$ and $|E(PG_n)| = 3n$. To prove that PG_n is sum divisor cordial, it suffices to construct a bijection $f : V(PG_n) \to \{1,2,\ldots,2n+1\}$ satisfying $|e_f(0) - e_f(1)| \leq 1$.

For all cases, we define $f(x) = 1$. Write $n = 4t+r$, where $t \in \mathbb{N}\cup\{0\}$ and $r \in \{0,1,2,3\}$. The labels for the remaining vertices x_i and y_i $(1 \leq i \leq n)$ are assigned according to the following cases.

Case 1. $r = 0$. For $1 \leq k \leq \frac{n}{4}$, we define the labels of x_i and y_i by

$$f(x_i) = \begin{cases} 8k-5, & i = 4k-3, \\ 8k-3, & i = 4k-2, \\ 8k-4, & i = 4k-1, \\ 8k-2, & i = 4k, \end{cases} \quad \text{and} \quad f(y_i) = \begin{cases} 8k-6, & i = 4k-3, \\ 8k-1, & i = 4k-2, \\ 8k+1, & i = 4k-1, \\ 8k, & i = 4k. \end{cases}$$

The induced labels for the edges incident to x are $f^*(x_1x) = 1$ and $f^*(y_1x) = 0$. For the remaining edges, the induced labels are given by

$$f^*(x_iy_i) = \begin{cases} 0, & \text{if } i \text{ is odd}, \\ 1, & \text{if } i \text{ is even}, \end{cases} \qquad (1 \leq i \leq n),$$

$$f^*(x_ix_{i+1}) = \begin{cases} 1, & \text{if } i \text{ is odd}, \\ 0, & \text{if } i \text{ is even}, \end{cases} \qquad (1 \leq i \leq n-1),$$

$$f^*(y_iy_{i+1}) = \begin{cases} 0, & \text{if } i \text{ is odd}, \\ 1, & \text{if } i \text{ is even}, \end{cases} \qquad (1 \leq i \leq n-1).$$

Counting the number of edges with labels 0 and 1 yields $e_f(0) = \frac{3n}{2}$ and $e_f(1) = \frac{3n}{2}$, which gives $|e_f(0) - e_f(1)| = 0$.

Case 2. $r = 1$. For $1 \leq i \leq n-1$ (which is omitted when $n = 1$), we define the labels of x_i and y_i as in Case 1. For $i = n$, we define

$$f(x_n) = 2n+1, \qquad f(y_n) = 2n.$$

The induced labels for the remaining edges associated with $i = n$ are $f^*(x_n y_n) = 0$, $f^*(x_{n-1}x_n) = 0$, and $f^*(y_{n-1}y_n) = 1$ (where the terms $x_{n-1}x_n$ and $y_{n-1}y_n$ are omitted when $n = 1$). Counting the edges with labels 0 and 1 yields $e_f(0) = \frac{3n+1}{2}$ and $e_f(1) = \frac{3n-1}{2}$, which gives $|e_f(0) - e_f(1)| = 1$.

Case 3. $r = 2$. For $1 \leq i \leq n-2$ (which is omitted when $n = 2$), we define the labels of x_i and y_i as in Case 1. For $i \in \{n-1, n\}$, we define

$$\begin{aligned} f(x_{n-1}) &= 2n-1, & f(y_{n-1}) &= 2n-2, \\ f(x_n) &= 2n+1, & f(y_n) &= 2n. \end{aligned}$$

The induced labels for the remaining edges are

$$\begin{aligned} f^*(x_{n-1}y_{n-1}) &= 0, & f^*(x_n y_n) &= 0, \\ f^*(x_{n-2}x_{n-1}) &= 0, & f^*(y_{n-2}y_{n-1}) &= 1, \\ f^*(x_{n-1}x_n) &= 1, & f^*(y_{n-1}y_n) &= 1, \end{aligned}$$

where the terms $x_{n-2}x_{n-1}$ and $y_{n-2}y_{n-1}$ are omitted when $n = 2$. Counting the edges with labels 0 and 1 yields $e_f(0) = \frac{3n}{2}$ and $e_f(1) = \frac{3n}{2}$, which gives $|e_f(0) - e_f(1)| = 0$.

Case 4. $r = 3$. For $1 \leq i \leq n-3$ (which is omitted when $n = 3$), we define the labels of x_i and y_i as in Case 1. For $i \in \{n-2, n-1, n\}$, we define

$$\begin{aligned} f(x_{n-2}) &= 2n-3, & f(y_{n-2}) &= 2n-4, \\ f(x_{n-1}) &= 2n-1, & f(y_{n-1}) &= 2n-2, \\ f(x_n) &= 2n+1, & f(y_n) &= 2n. \end{aligned}$$

The induced labels for the remaining edges are

$$\begin{aligned} f^*(x_{n-2}y_{n-2}) &= 0, & f^*(x_{n-2}x_{n-1}) &= 1, & f^*(y_{n-2}y_{n-1}) &= 1, \\ f^*(x_{n-1}y_{n-1}) &= 0, & f^*(x_{n-1}x_n) &= 1, & f^*(y_{n-1}y_n) &= 1, \\ f^*(x_n y_n) &= 0, & f^*(x_{n-3}x_{n-2}) &= 0, & f^*(y_{n-3}y_{n-2}) &= 1, \end{aligned}$$

where the terms $x_{n-3}x_{n-2}$ and $y_{n-3}y_{n-2}$ are omitted when $n = 3$. Counting the edges with labels 0 and 1 yields $e_f(0) = \frac{3n-1}{2}$ and $e_f(1) = \frac{3n+1}{2}$, which gives $|e_f(0) - e_f(1)| = 1$.

Therefore, in all cases, $|e_f(0) - e_f(1)| \leq 1$. By Definition 1.1, PG_n is a sum divisor

cordial graph for all $n \in \mathbb{N}$. ■

Example 2.2. Sum divisor cordial labeling of PG_n for $n = 4, 5, 6, 7$, where solid and dashed lines represent edges labeled 1 and 0, respectively.

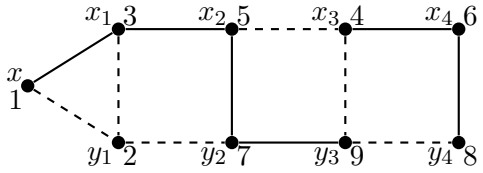

Figure 1: The graph PG_4 $(r = 0)$.

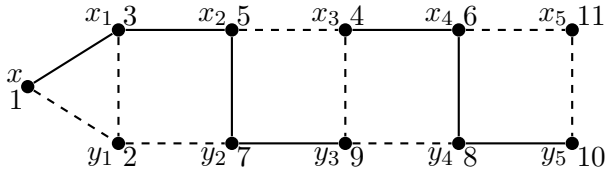

Figure 2: The graph PG_5 $(r = 1)$.

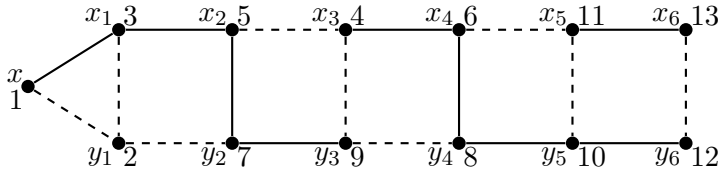

Figure 3: The graph PG_6 $(r = 2)$.

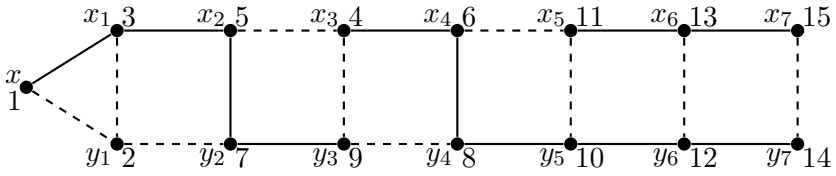

Figure 4: The graph PG_7 $(r = 3)$.

Theorem 2.3. *The diamond ladder graph DL_n is a sum divisor cordial graph for all $n \in \mathbb{N}$.*

Proof: By Definition 1.3, the diamond ladder graph DL_n has $|V(DL_n)| = 4n$ and $|E(DL_n)| = 8n - 3$. To prove that DL_n is sum divisor cordial, it suffices to construct a bijection $f : V(DL_n) \to \{1, 2, \ldots, 4n\}$ satisfying $|e_f(0) - e_f(1)| \leq 1$. We define the labeling f for $1 \leq i \leq n$ by

$$f(x_i) = 4i - 3, \qquad f(y_i) = 4i - 2,$$

$$f(z_{2i-1}) = 4i - 1, \qquad f(z_{2i}) = 4i.$$

The induced labels for each edge set are given by

$$\begin{aligned}
f^*(x_i y_i) &= 0, \qquad (1 \le i \le n),\\
f^*(x_i z_{2i-1}) &= 1, \qquad (1 \le i \le n),\\
f^*(y_i z_{2i-1}) &= 0, \qquad (1 \le i \le n),\\
f^*(x_i z_{2i}) &= 0, \qquad (1 \le i \le n),\\
f^*(y_i z_{2i}) &= 1, \qquad (1 \le i \le n),\\
f^*(x_i x_{i+1}) &= 1, \qquad (1 \le i \le n-1),\\
f^*(y_i y_{i+1}) &= 1, \qquad (1 \le i \le n-1),\\
f^*(z_{2i} z_{2i+1}) &= 0, \qquad (1 \le i \le n-1).
\end{aligned}$$

Counting the number of edges with labels 0 and 1 yields

$$\begin{aligned}
e_f(0) &= n + n + n + (n-1) = 4n - 1,\\
e_f(1) &= n + n + (n-1) + (n-1) = 4n - 2.
\end{aligned}$$

Consequently, $|e_f(0) - e_f(1)| = |(4n-1) - (4n-2)| = 1 \le 1$, which completes the proof. ∎

Example 2.4. Sum divisor cordial labeling of DL_n for $n = 1, 2, 3$, where solid and dashed lines represent edges labeled 1 and 0, respectively.

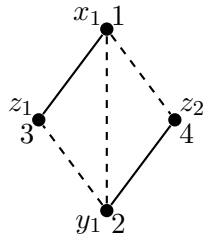

Figure 5: The graph DL_1.

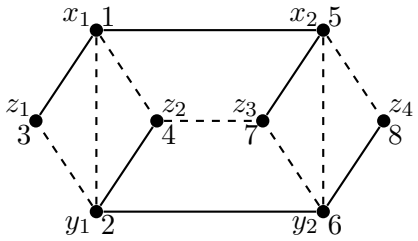

Figure 6: The graph DL_2.

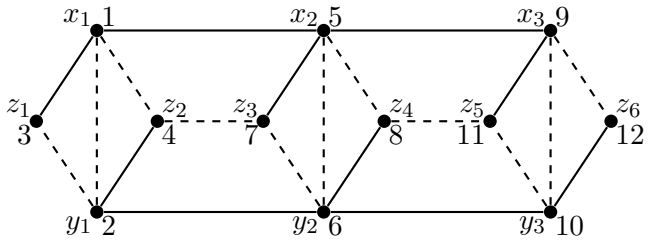

Figure 7: The graph DL_3.

Theorem 2.5. *The graph D_n^* is a sum divisor cordial graph for all $n \in \mathbb{N}$.*

Proof: By Definition 1.4, the graph D_n^* has $|V(D_n^*)| = 4n$ and $|E(D_n^*)| = 7n - 3$. To prove that D_n^* is sum divisor cordial, it suffices to construct a bijection $f : V(D_n^*) \to \{1, 2, \ldots, 4n\}$ satisfying $|e_f(0) - e_f(1)| \leq 1$. We consider two cases based on the parity of n.

Case 1. n is even. For $1 \leq j \leq \frac{n}{2}$, we define the bijection f explicitly by

$$\begin{aligned} f(x_{2j-1}) &= 8j-6, & f(x_{2j}) &= 8j-5,\\ f(y_{2j-1}) &= 8j-4, & f(y_{2j}) &= 8j,\\ f(z_{4j-3}) &= 8j-2, & f(z_{4j-1}) &= 8j-3,\\ f(z_{4j-2}) &= 8j-7, & f(z_{4j}) &= 8j-1. \end{aligned}$$

The induced labels for each edge set are given by

$$\begin{aligned} f^*(x_i z_{2i-1}) &= 1, \quad \text{for } 1 \leq i \leq n,\\ f^*(y_i z_{2i-1}) &= \begin{cases} 1, & \text{for odd } i,\\ 0, & \text{for even } i, \end{cases}\\ f^*(x_i z_{2i}) &= \begin{cases} 0, & \text{for odd } i,\\ 1, & \text{for even } i, \end{cases}\\ f^*(y_i z_{2i}) &= 0, \quad \text{for } 1 \leq i \leq n,\\ f^*(x_i x_{i+1}) &= 0, \quad \text{for } 1 \leq i \leq n-1,\\ f^*(y_i y_{i+1}) &= 1, \quad \text{for } 1 \leq i \leq n-1,\\ f^*(z_{2i} z_{2i+1}) &= \begin{cases} 1, & \text{for odd } i \leq n-1,\\ 0, & \text{for even } i \leq n-1. \end{cases} \end{aligned}$$

Since there are exactly $\frac{n}{2}$ odd indices and $\frac{n}{2}$ even indices in $\{1, 2, \ldots, n\}$, counting the edges with label 1 yields $e_f(1) = n + \frac{n}{2} + \frac{n}{2} + 0 + 0 + (n-1) + \frac{n}{2} = \frac{7n-2}{2}$. Since $|E(D_n^*)| = 7n-3$, we obtain $e_f(0) = (7n-3) - \frac{7n-2}{2} = \frac{7n-4}{2}$, which gives $|e_f(0) - e_f(1)| = 1$.

Case 2. n is odd. For $1 \leq j \leq \frac{n-1}{2}$ (which is omitted when $n = 1$), we define the labels

for $1 \leq i \leq n-1$ identically to the even case. For $i = n$, we define

$$f(x_n) = 4n-2, \qquad f(y_n) = 4n-3,$$
$$f(z_{2n-1}) = 4n-1, \qquad f(z_{2n}) = 4n.$$

The induced labels for the edges associated with $i = n$ are $f^*(x_n z_{2n-1}) = 0$, $f^*(y_n z_{2n-1}) = 1$, $f^*(x_n z_{2n}) = 1$, and $f^*(y_n z_{2n}) = 0$. For the edges between $i = n-1$ and $i = n$, we have $f^*(x_{n-1}x_n) = 0$, $f^*(y_{n-1}y_n) = 0$, and $f^*(z_{2n-2}z_{2n-1}) = 1$ (where these three terms are omitted when $n = 1$). Combining these with the edge counts for $1 \leq i \leq n-1$, we obtain

$$e_f(1) = 4\left(\frac{n-1}{2}\right) + 2 + 3\left(\frac{n-1}{2}\right) = \frac{7n-3}{2}.$$

Since $|E(D_n^*)| = 7n-3$, it follows that $e_f(0) = (7n-3) - \frac{7n-3}{2} = \frac{7n-3}{2}$, which gives $|e_f(0) - e_f(1)| = 0$.

Therefore, in all cases, $|e_f(0) - e_f(1)| \leq 1$. By Definition 1.1, D_n^* is a sum divisor cordial graph for all $n \in \mathbb{N}$. ■

Example 2.6. Sum divisor cordial labeling of D_n^* for $n = 1, 2, 3$, where solid and dashed lines represent edges labeled 1 and 0, respectively.

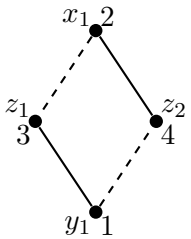

Figure 8: The graph D_1^* (n is odd).

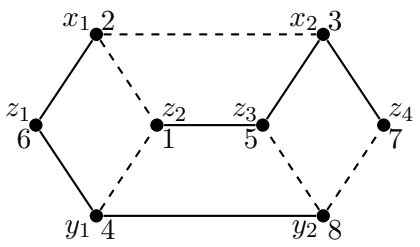

Figure 9: The graph D_2^* (n is even).

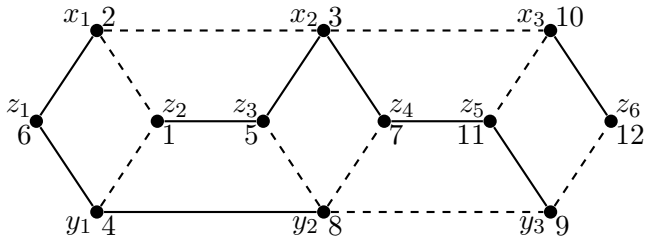

Figure 10: The graph D_3^* (n is odd).

Theorem 2.7. *The jellyfish graph $J(m,n)$ is a sum divisor cordial graph for all $m,n \in \mathbb{N}$.*

Proof: Let $V(J(m,n)) = \{v_1,v_2,v_3,v_4\} \cup X \cup Y$, where $X = \{x_1,x_2,\ldots,x_m\}$ and $Y = \{y_1,y_2,\ldots,y_n\}$ denote the sets of pendant vertices attached to v_2 and v_4, respectively. The edge set of $J(m,n)$ is given by

$$E(J(m,n)) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_1, v_1v_3\} \cup \{v_2x_i \mid 1 \le i \le m\} \cup \{v_4y_j \mid 1 \le j \le n\}.$$

Thus, $|V(J(m,n))| = m+n+4$ and $|E(J(m,n))| = m+n+5$.

To prove that $J(m,n)$ is sum divisor cordial, we construct a bijection $f : V(J(m,n)) \to \{1,2,\ldots,m+n+4\}$ such that $|e_f(0)-e_f(1)| \le 1$. We define f by assigning explicit labels to the vertices v_1, v_2, v_3, and v_4, and distributing the remaining even and odd labels among the pendant vertices. We consider four cases depending on whether m and n are even or odd.

Case 1. Both m and n are even. We define the labels of v_1, v_2, v_3, and v_4 as

$$f(v_1) = 2, \qquad f(v_2) = 4,$$
$$f(v_3) = 1, \qquad f(v_4) = 3.$$

For the pendant vertices, we assign the remaining labels explicitly by defining

$$f(x_i) = \begin{cases} 2i+4, & 1 \le i \le \frac{m}{2}, \\ 2i-m+3, & \frac{m}{2}+1 \le i \le m, \end{cases}$$
$$f(y_j) = \begin{cases} m+2j+3, & 1 \le j \le \frac{n}{2}, \\ m-n+2j+4, & \frac{n}{2}+1 \le j \le n. \end{cases}$$

Among the edges $\{v_1v_2, v_2v_3, v_3v_4, v_4v_1, v_1v_3\}$, we have $f^*(v_1v_2) = 1$, $f^*(v_3v_4) = 1$, and 0 for the remaining three edges. For the pendant edges, $f(x_i)$ is even for $1 \le i \le \frac{m}{2}$, which yields $f^*(v_2x_i) = 1$ for $\frac{m}{2}$ edges. Similarly, $f(y_j)$ is odd for $1 \le j \le \frac{n}{2}$, which yields $f^*(v_4y_j) = 1$ for $\frac{n}{2}$ edges. Thus, $e_f(1) = 2+\frac{m}{2}+\frac{n}{2} = \frac{m+n+4}{2}$. Since $|E(J(m,n))| = m+n+5$, it follows that $e_f(0) = \frac{m+n+6}{2}$, which gives $|e_f(0) - e_f(1)| = 1$.

Case 2. Both m and n are odd. We define the labels of v_1, v_2, v_3, and v_4 as

$$f(v_1) = 2, \qquad f(v_2) = 4,$$
$$f(v_3) = 1, \qquad f(v_4) = 3.$$

For the pendant vertices, we assign the remaining labels explicitly by defining

$$f(x_i) = \begin{cases} 2i+4, & 1 \le i \le \frac{m+1}{2}, \\ 2i-m+2, & \frac{m+3}{2} \le i \le m, \end{cases}$$
$$f(y_j) = \begin{cases} m+2j+2, & 1 \le j \le \frac{n+1}{2}, \\ m-n+2j+4, & \frac{n+3}{2} \le j \le n, \end{cases}$$

where the second conditions of $f(x_i)$ and $f(y_j)$ are omitted when $m = 1$ and $n = 1$, respectively. Among the edges $\{v_1v_2, v_2v_3, v_3v_4, v_4v_1, v_1v_3\}$, we have $f^*(v_1v_2) = 1$, $f^*(v_3v_4) = 1$, and 0 for the remaining three edges. For the pendant edges, $f(x_i)$ is even for $1 \le i \le \frac{m+1}{2}$, which yields $f^*(v_2x_i) = 1$ for $\frac{m+1}{2}$ edges. Similarly, $f(y_j)$ is odd for $1 \le j \le \frac{n+1}{2}$, which yields $f^*(v_4y_j) = 1$ for $\frac{n+1}{2}$ edges. Thus, $e_f(1) = 2 + \frac{m+1}{2} + \frac{n+1}{2} = \frac{m+n+6}{2}$. Since $|E(J(m,n))| = m+n+5$, it follows that $e_f(0) = \frac{m+n+4}{2}$, which gives $|e_f(0) - e_f(1)| = 1$.

Case 3. m is even and n is odd. We define the labels of v_1, v_2, v_3, and v_4 as

$$f(v_1) = 2, \qquad f(v_2) = 4,$$
$$f(v_3) = 1, \qquad f(v_4) = 3.$$

For the pendant vertices, we assign the remaining labels explicitly by defining

$$f(x_i) = \begin{cases} 2i+4, & 1 \le i \le \frac{m}{2}, \\ 2i-m+3, & \frac{m}{2}+1 \le i \le m, \end{cases}$$
$$f(y_j) = \begin{cases} m+2j+3, & 1 \le j \le \frac{n+1}{2}, \\ m-n+2j+3, & \frac{n+3}{2} \le j \le n, \end{cases}$$

where the second condition of $f(y_j)$ is omitted when $n = 1$.
Among the edges $\{v_1v_2, v_2v_3, v_3v_4, v_4v_1, v_1v_3\}$, we have $f^*(v_1v_2) = 1$, $f^*(v_3v_4) = 1$, and 0 for the remaining three edges. For the pendant edges, $f(x_i)$ is even for $1 \le i \le \frac{m}{2}$, which yields $f^*(v_2x_i) = 1$ for $\frac{m}{2}$ edges. Similarly, $f(y_j)$ is odd for $1 \le j \le \frac{n+1}{2}$, which yields $f^*(v_4y_j) = 1$ for $\frac{n+1}{2}$ edges. Thus, $e_f(1) = 2 + \frac{m}{2} + \frac{n+1}{2} = \frac{m+n+5}{2}$. Since $|E(J(m,n))| = m+n+5$, it follows that $e_f(0) = \frac{m+n+5}{2}$, which gives $|e_f(0) - e_f(1)| = 0$.

Case 4. m is odd and n is even. We define the labels of v_1, v_2, v_3, and v_4 as

$$f(v_1) = 1, \qquad f(v_2) = 3,$$

$$f(v_3) = 2, \qquad f(v_4) = 4.$$

For the pendant vertices, we assign the remaining labels explicitly by defining

$$f(x_i) = \begin{cases} 2i+3, & 1 \le i \le \frac{m+1}{2}, \\ 2i-m+3, & \frac{m+3}{2} \le i \le m, \end{cases}$$

$$f(y_j) = \begin{cases} m+2j+3, & 1 \le j \le \frac{n}{2}, \\ m-n+2j+4, & \frac{n}{2}+1 \le j \le n, \end{cases}$$

where the second condition of $f(x_i)$ is omitted when $m = 1$. Among the edges $\{v_1v_2, v_2v_3, v_3v_4, v_4v_1, v_1v_3\}$, we have $f^*(v_1v_2) = 1$, $f^*(v_3v_4) = 1$, and 0 for the remaining three edges. For the pendant edges, $f(x_i)$ is odd for $1 \le i \le \frac{m+1}{2}$, which yields $f^*(v_2x_i) = 1$ for $\frac{m+1}{2}$ edges. Similarly, $f(y_j)$ is even for $1 \le j \le \frac{n}{2}$, which yields $f^*(v_4y_j) = 1$ for $\frac{n}{2}$ edges. Thus, $e_f(1) = 2 + \frac{m+1}{2} + \frac{n}{2} = \frac{m+n+5}{2}$. Since $|E(J(m,n))| = m+n+5$, it follows that $e_f(0) = \frac{m+n+5}{2}$, which gives $|e_f(0) - e_f(1)| = 0$.

Therefore, in all cases, $|e_f(0) - e_f(1)| \le 1$. By Definition 1.1, $J(m,n)$ is a sum divisor cordial graph for all $m, n \in \mathbb{N}$. ■

Example 2.8. Sum divisor cordial labeling of $J(m,n)$ for $m, n \in \{2,3\}$, where solid and dashed lines represent edges labeled 1 and 0, respectively.

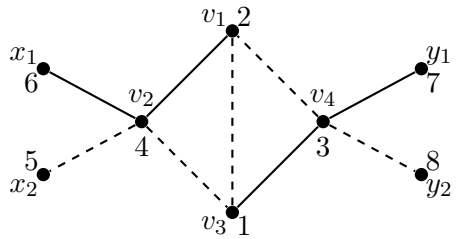

Figure 11: The graph $J(2,2)$ (both m and n are even).

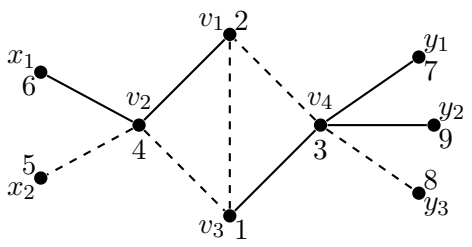

Figure 12: The graph $J(2,3)$ (m is even and n is odd).

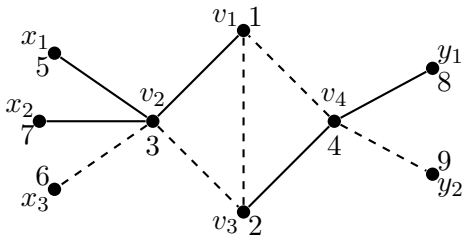

Figure 13: The graph $J(3,2)$ (m is odd and n is even).

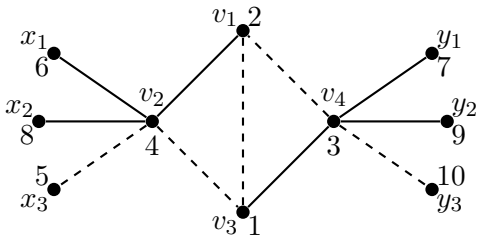

Figure 14: The graph $J(3,3)$ (both m and n are odd).

3 Concluding Remarks

In this work, sum divisor cordial labelings of the pagoda graph PG_n, the diamond ladder graph DL_n, the graph D_n^*, and the jellyfish graph $J(m,n)$ were investigated by providing explicit labeling constructions for each family. Specifically, we proved that PG_n, DL_n, and D_n^* are sum divisor cordial graphs for all positive integers n, and established that $J(m,n)$ is a sum divisor cordial graph for all positive integers m and n. These results may serve as a basis for further investigations of sum divisor cordial labeling for other classes of graphs and related graph operations.

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