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Distance Version of F-index of Some Graph Operations

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Abstract

The distance version of F- index of a graph is defined and exact values of this index have been found for some graph operations.

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1 Introduction

A topological index of a graph is a numerical quantity which is structural invariant. Means that it is fixed under graph automorphism. The simplest topological indices are the number of vertices and edges of a graph. In chemistry, biochemistry and nanotechnology different topological indices are found to be useful in isomer discrimination, structure-property relationship, structure-activity relationship and pharmaceutical drug design.

All graphs considered are simple and connected graphs. We denote the vertex and the edge set of a graph G by $V(G)$ and $E(G)$, respectively. $d_G(v)$ denotes the degree of a vertex v in G. The number of elements in the vertex set of a graph G is called the order

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of G and is denoted by $v(G)$. The number of elements in the edge set of a graph G is called the size of G and is denoted by $e(G)$. A graph with order n and size m edges is called a (n,m)-graph. For any $u, v \in V(G)$, the distance between u and v in G, denoted by $d_G(u,v)$, is the length of a shortest (u,v)-path in G. The edge connecting the vertices u and v will be denoted by uv. The complement of a graph G is denoted by $\overline{G}$.

The join of graphs G_1 and G_2 is denoted by G_1+G_2, and it is the graph with vertex set $V(G_1)\cup V(G_2)$ and the edge set $E(G_1+G_2) = E(G_1)\cup E(G_2)\cup \{u_1u_2|u_1 \in V(G_1), u_2 \in V(G_2)\}$. The composition of graphs G_1 and G_2 is denoted by $G_1[G_2]$, and it is the graph with vertex set $V(G_1)\times V(G_2)$, and two vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are adjacent if (u_1 is adjacent to v_1) or ($u_1 = v_1$ and u_2 and v_2 are adjacent). The disjunction of graphs G_1 and G_2 is denoted by $G_1 \vee G_2$, and it is the graph with vertex set $V(G_1) \times V(G_2)$ and $E(G_1 \vee G_2) = \{(u_1, u_2)(v_1, v_2)|u_1v_1 \in E(G_1)$ or $u_2v_2 \in E(G_2)\}$. The symmetric difference of graphs G_1 and G_2 is denoted by $G_1 \oplus G_2$, and it is the graph with vertex set $V(G_1)\times V(G_2)$ and edge set $E(G_1\oplus G_2) = \{(u_1, u_2)(v_1, v_2)|u_1v_1 \in E(G_1)$ or $u_2v_2 \in E(G_2)$ but not both $\}$. The Cartesian product of G_1 and G_2, denoted by $G_1\Box G_2$, is the graph vertex set $V(G_1) \times V(G_2)$ and any two vertices (u_p, v_s) and (u_q, v_s) are adjacent if and only if $[u_p = u_q$ and $v_rv_s \in E(G_2)]$ or $[v_r = v_s$ and $u_pu_q \in E(G_1)]$.

Let G be a connected graph. The Wiener index $W(G)$ of a graph G is defined as

$$W(G) = \sum_{\{u,v\}\subseteq V(G)} d_G(u,v) = \frac{1}{2}\sum_{u,v\in V(G)} d_G(u,v).$$

Dobrynin and Kochetova[9] and Gutman[13] independently proposed a vertex -degree-Weighted version of Wiener index called degree distance or Schultz molecular topological index, which is defined for a connected graph G as

$$DD(G) = \sum_{\{u,v\}\subseteq V(G)} d_G(u,v)[d_G(u)+d_G(v)] = \frac{1}{2}\sum_{u,v\in V(G)} d_G(u,v)[d_G(u)+d_G(v)].$$

The Zagreb indices have been introduced more than thirty years ago by Gutman and Trianjestic [14]. The first Zagreb index $M_1(G)$ of a graph G is defined as

$$M_1(G) = \sum_{uv\in E(G)} [d_G(u)+d_G(v)] = \sum_{v\in V(G)} d_G^2(v).$$

The second Zagreb index $M_2(G)$ of a graph G is defined as

$$M_2(G) = \sum_{uv \in E(G)} d_G(u) d_G(v).$$

The first Zagreb coindex $\overline{M}_1(G)$ of a graph G is defined as

$$\overline{M}_1(G) = \sum_{uv \notin E(G)} [d_G(u) + d_G(v)].$$

The second Zagreb coindex $\overline{M}_2(G)$ of a graph G is defined as

$$\overline{M}_2(G) = \sum_{uv \notin E(G)} d_G(u) d_G(v).$$

The Zagreb indices are found to have applications in QSPR and QSAR studies as well, see [8].

These indices were introduced in a paper 1972[1], within a study of the structure-dependency of total Π-electron energy (ε). It was shown that ε depends on the sum of squares of the vertex degrees of the molecular graph (later named first zagreb index), and thus provides a measure of the branching of the carbon-atom skeleton. In the same paper, also the sum of cubes of degrees of vertices of the molecular graph was found to influence ε, but this topological index was never again investigated and was left to oblivion. However this index was not further studied till then, except in a recent article by Furtula and Gutman [12] where they reinvestigated this index and studied some basic properties of this index. They showed that the predictive ability of this index is almost similar to that of first Zagreb index and for the entropy and acentric factor, both of them yield correlation coefficients greater than 0.95. In [17], this index as "forgotten topological index" or "F-index", which is defined for a graph G as

$$F(G) = \sum_{v \in V(G)} d_G(v)^3 = \sum_{uv \in E(G)} [d_G(u)^2 + d_G(v)^2].$$

As we know that some chemically interesting graphs can be found by different graph operations on some general or particular graphs, it is important to study such graph operations in order to understand how it is related to the corresponding topological indices of the original graphs. In [15], Khalifeh et. al, derived some exact formulae for computing first and second Zagreb indices under some graph operations. In [5], Das et. al. derived some upper bounds for multiplicative Zagreb indices for different graph operatins. In [7], De et.al, authors computed some bounds and exact formulae of the connective eccentric index under different graphs oprations. There are several other results regarding various topological indices under different graph operatins are available in the literature. In [2], Azari and Inranmanesh presented explicit formulae for computing the eccentric-distance sum of different graph operations. Interested readers are referred to [3, 4, 6, 11, 16, 18, 19] in this regard. In this paper, we define new index as the distance

version of F- index and denoted by $DF(G)$, so that

$$\begin{aligned} DF(G) &= \sum_{\{u,v\}\subseteq V(G)} d_G(u,v)[d_G(u)^2 + d_G(v)^2] \\ &= \frac{1}{2}\sum_{u,v\in V(G)} d_G(u,v)[d_G(u)^2 + d_G(v)^2]. \end{aligned}$$

In this paper, we present some exact expressions for the distance version of F-index of different graph operations such as join, composition, disjunction, Cartesian product and symmetric difference of two graphs.

2 Main Results and Discussions

2.1 Basic Lemmas

Lemma 2.1. [20] Let G_1 and G_2 be two simple connected graphs. The number of vertices and edges of graph G_i is denoted by n_i and e_i respectively for $i = 1, 2$. Then we have (i)

$$d_{G_1+G_2}(u,v) = \begin{cases} 1, & uv \in E(G_1) \text{ or } uv \in E(G_2) \text{ or } (u \in V(G_1) \text{ and } v \in V(G_2)) \\ 2, & \text{otherwise} \end{cases}$$

For a vertex u of G_1, $d_{G_1+G_2}(u) = d_{G_1}(u) + n_2$, and for a vertex v of $G_2, d_{G_1+G_2}(v) = d_{G_2}(v) + n_1$.

(ii)

$$d_{G_1[G_2]}((u_1,v_1),(u_2,v_2)) = \begin{cases} d_{G_1}(u_1,u_2), & u_1 \neq u_2 \\ 1, & u_1 = u_2, v_1v_2 \in E(G_2) \\ 2, & \text{otherwise} \end{cases}$$

$d_{G_1[G_2]}(u,v) = n_2 d_{G_1}(u) + d_{G_2}(v)$.
(iii)

$$d_{G_1 \vee G_2}((u_1,v_1),(u_2,v_2)) = \begin{cases} 1, & u_1u_2 \in E(G_1) \text{ or } v_1v_2 \in E(G_2) \\ 2, & \text{otherwise} \end{cases}$$

$d_{G_1 \vee G_2}((u,v)) = n_2 d_{G_1}(u) + n_1 d_{G_2}(v) - d_{G_1}(u) d_{G_2}(v)$.
(iv)

$$d_{G_1 \oplus G_2}((u_1,v_1),(u_2,v_2)) = \begin{cases} 1, & u_1u_2 \in E(G_1) \text{ or } v_1v_2 \in E(G_2) \text{ but not both} \\ 2, & \text{otherwise} \end{cases}$$

$d_{G_1 \oplus G_2}((u,v)) = n_2 d_{G_1}(u) + n_1 d_{G_2}(v) - 2d_{G_1}(u) d_{G_2}(v)$.

Remark 2.2. For a graph G, let $A(G) = \{(x,y) \in V(G) \times V(G) \mid x$ and y are adjacent in $G\}$ and let $B(G) = \{(x,y) \in V(G) \times V(G) \mid x$ and y are not adjacent in $G\}$. For each $x \in V(G)$, $(x,x) \in B(G)$. Clearly, $A(G) \cup B(G) = V(G) \times V(G)$. Let $C(G) = \{(x,x) \mid x \in V(G)\}$ and $D(G) = B(G) - C(G)$. Clearly $B(G) = C(G) \cup D(G)$, $C(G) \cap D(G) = \emptyset$. The summation $\sum\limits_{(x,y) \in A(G)}$ runs over the ordered pairs of $A(G)$. For simplicity, we write the summation $\sum\limits_{(x,y) \in A(G)}$ as $\sum\limits_{xy \in G}$. Similarly, we write the summation $\sum\limits_{(x,y) \in B(G)}$ as $\sum\limits_{xy \notin G}$. Also the summation $\sum\limits_{xy \in E(G)}$ runs over the edges of G. We denote the summation $\sum\limits_{x,y \in V(G)}$ by $\sum\limits_{x,y \in G}$. The summation $\sum\limits_{(x,y) \in D(G)}$ eqivalent to $\sum\limits_{xy \notin G, x \neq y}$ and similarly the summation $\sum\limits_{(x,y) \in C(G)}$ eqivalent to $\sum\limits_{xy \notin G, x = y}$.

Lemma 2.3. [10]

$$\sum_{xy \in G} d_G(x) = M_1(G)$$

Lemma 2.4.

$$\sum_{xy \notin G} 1 = 2e(\overline{G}) + v(G)$$

Proof:

$$\begin{aligned}\sum_{xy\notin G} 1 &= \sum_{(x,y)\in D(G)} 1 + \sum_{(x,x)\in C(G)} 1\\ &= 2e(\overline{G}) + v(G).\end{aligned}$$

Hence proved the lemma. ∎

Lemma 2.5.

$$\sum_{xy\notin G} d_G(x) = 2e(\overline{G})(v(G)-1) + 2e(G) - M_1(\overline{G})$$

Proof:

$$\begin{aligned}\sum_{xy\notin G} d_G(x) &= \sum_{(x,y)\in D(G)} d_G(x) + \sum_{(x,x)\in C(G)} d_G(x)\\ &= \sum_{(x,y)\in D(G)} \left\{v(G) - 1 - d_{\overline{G}}(x)\right\} + \sum_{(x,x)\in C(G)} d_G(x)\\ &= (v(G)-1)\sum_{(x,y)\in D(G)} 1 - \sum_{(x,y)\in D(G)} d_{\overline{G}}(x) + 2e(G)\\ &= (v(G)-1)2e(\overline{G}) - \sum_{(x,y)\in A(\overline{G})} d^2_{\overline{G}}(x) + 2e(G)\\ &= (v(G)-1)2e(\overline{G}) - \sum_{xy\in \overline{G}} d^2_{\overline{G}}(x) + 2e(G)\\ &= 2e(\overline{G})(v(G)-1) + 2e(G) - M_1(\overline{G})\end{aligned}$$

Hence proved the lemma. ∎

Lemma 2.6.

$$\sum_{xy\notin G} d_G(x)d_G(y) = 2\overline{M}_2(G) + M_1(G)$$

Proof:

$$\begin{aligned}\sum_{xy\notin G} d_G(x)d_G(y) &= \sum_{(x,y)\in D(G)} d_G(x)d_G(y) + \sum_{(x,x)\in C(G)} d_G(x)d_G(x)\\ &= 2\sum_{xy\notin E(G)} d_G(x)d_G(y) + \sum_{x\in V(G)} d^2_G(x)\end{aligned}$$

$$= 2\overline{M}_2(G) + M_1(G)$$

Hence proved the lemma. ∎

Lemma 2.7.

$$\sum_{xy \notin G} [d_G(x) + d_G(y)] = 2\overline{M}_1(G) + 4e(G)$$

Proof:

$$\begin{aligned}
\sum_{xy \notin G} [d_G(x) + d_G(y)] &= \sum_{(x,y) \in C(G)} [d_G(x) + d_G(y)] + \sum_{(x,y) \in D(G)} [d_G(x) + d_G(y)] \\
&= \sum_{x \in V(G)} 2d_G(x) + 2 \sum_{xy \notin E(G)} [d_G(x) + d_G(y)] \\
&= 4e(G) + 2\overline{M}_1(G)
\end{aligned}$$

Hence proved the lemma. ∎

Lemma 2.8.

$$\sum_{xy \notin G, x \neq y} d_G^2(x) = (v(G) - 1)M_1(G) - F(G)$$

Proof:

$$\begin{aligned}
\sum_{xy \notin G, x \neq y} d_G^2(x) &= \sum_{x \in V(G)} [v(G) - 1 - d_G(x)]d_G^2(x) \\
&= (v(G) - 1) \sum_{x \in V(G)} d_G^2(x) - \sum_{x \in V(G)} d_G^3(x) \\
&= (v(G) - 1)M_1(G) - F(G)
\end{aligned}$$

Hence proved the lemma. ∎

Lemma 2.9.

$$\sum_{xy \notin G, x \neq y} d_G(x) = (v(G) - 1)2e(G) - M_1(G)$$

Proof:

$$\sum_{xy \notin G, x \neq y} d_G(x) = \sum_{x \in V(G)} [v(G) - 1 - d_G(x)]d_G(x)$$

$$
\begin{aligned}
&= (v(G)-1)\sum_{x\in V(G)} d_G(x) - \sum_{x\in V(G)} d_G^2(x)\\
&= (v(G)-1)2e(G) - M_1(G)
\end{aligned}
$$

Hence proved the lemma. ∎

Lemma 2.10.

$$\sum_{xy\notin G, x\neq y} 1 = (v(G)-1)v(G) - 2e(G)$$

Proof:

$$
\begin{aligned}
\sum_{xy\notin G, x\neq y} 1 &= \sum_{x\in V(G)} [v(G)-1-d_G(x)]\\
&= (v(G)-1)\sum_{x\in V(G)} 1 - \sum_{x\in V(G)} d_G(x)\\
&= (v(G)-1)v(G) - 2e(G)
\end{aligned}
$$

Hence proved the lemma. ∎

Lemma 2.11.

$$\sum_{xy\notin G} d_G^2(x) = (v(G)-1)M_1(G) - F(G) + M_1(G)$$

Proof:

$$
\begin{aligned}
\sum_{xy\notin G} d_G^2(x) &= \sum_{xy\notin G, x\neq y} d_G^2(x) + \sum_{xy\notin G, x=y} d_G^2(x)\\
&= (v(G)-1)M_1(G) - F(G) + \sum_{x\in V(G)} d_G^2(x)\\
&= (v(G)-1)M_1(G) - F(G) + M_1(G)
\end{aligned}
$$

Hence proved the lemma. ∎

The Zagreb indices and Zagreb coindices, in particular the above Lemmas which are proved, will be helpful in presenting our main results in a compact form.

2.2 Join

In the following theorem, we compute the distance version of F- index of the join of two graphs.

Theorem 2.12. Let $G_i, i = 1, 2$, be a (n_i, m_i)- graph, then

$$\begin{aligned} 2 \times DF(G_1 + G_2) &= 2F(G_1) + 4n_2M_1(G_1) + 4n_2^2m_1 + 4[M_1(G_1)(n_1 - 1) - F(G_1)] \\ &+ 8n_2[(n_1 - 1)2m_1 - M_1(G_1)] + 4n_2^2[(n_1 - 1)n_1 - 2m_1] \\ &+ 2M_1(G_1)n_2 + 2n_2^3n_1 + 8n_2^2m_1 + 2M_1(G_2)n_1 + 2n_1^3n_2 + 8n_1^2m_2 \\ &+ 2F(G_2) + 4n_1M_1(G_2) + 4n_1^2m_2 + 4[M_1(G_2)(n_2 - 1) - F(G_2)] \\ &+ 8n_1[(n_2 - 1)2m_2 - M_1(G_2)] + 4n_1^2[(n_2 - 1)n_2 - 2m_2] \end{aligned}$$

Proof:

$$\begin{aligned} 2 \times DF(G_1 + G_2) &= \sum_{x \in V(G_1+G_2)} \sum_{y \in V(G_1+G_2)} d_{G_1+G_2}(x, y)\Big[d_{G_1+G_2}(x)^2 + d_{G_1+G_2}(y)^2\Big] \\ &= \sum_{x \in V(G_1+G_2)} \Big\{ \sum_{y \in V(G_1)} d_{G_1+G_2}(x, y)\Big[d_{G_1+G_2}(x)^2 + d_{G_1+G_2}(y)^2\Big] \\ &+ \sum_{y \in V(G_2)} d_{G_1+G_2}(x, y)\Big[d_{G_1+G_2}(x)^2 + d_{G_1+G_2}(y)^2\Big]\Big\} \\ &= \sum_{x \in V(G_1+G_2)} \sum_{y \in V(G_1)} d_{G_1+G_2}(x, y)\Big[d_{G_1+G_2}(x)^2 + d_{G_1+G_2}(y)^2\Big] \\ &+ \sum_{x \in V(G_1+G_2)} \sum_{y \in V(G_2)} d_{G_1+G_2}(x, y)\Big[d_{G_1+G_2}(x)^2 + d_{G_1+G_2}(y)^2\Big] \\ &= \sum_{x \in V(G_1)} \sum_{y \in V(G_1)} d_{G_1+G_2}(x, y)\Big[d_{G_1+G_2}(x)^2 + d_{G_1+G_2}(y)^2\Big] \end{aligned}$$

$$\begin{aligned} &+ \sum_{x \in V(G_2)} \sum_{y \in V(G_1)} d_{G_1+G_2}(x, y)\Big[d_{G_1+G_2}(x)^2 + d_{G_1+G_2}(y)^2\Big] \\ &+ \sum_{x \in V(G_1)} \sum_{y \in V(G_2)} d_{G_1+G_2}(x, y)\Big[d_{G_1+G_2}(x)^2 + d_{G_1+G_2}(y)^2\Big] \\ &+ \sum_{x \in V(G_2)} \sum_{y \in V(G_2)} d_{G_1+G_2}(x, y)\Big[d_{G_1+G_2}(x)^2 + d_{G_1+G_2}(y)^2\Big] \\ &= \sum_{x \in V(G_1)} \sum_{y \in V(G_1)} d_{G_1+G_2}(x, y)\Big[d_{G_1+G_2}(x)^2 + d_{G_1+G_2}(y)^2\Big] \\ &+ 2\sum_{x \in V(G_1)} \sum_{y \in V(G_2)} d_{G_1+G_2}(x, y)\Big[d_{G_1+G_2}(x)^2 + d_{G_1+G_2}(y)^2\Big] \\ &+ \sum_{x \in V(G_2)} \sum_{y \in V(G_2)} d_{G_1+G_2}(x, y)\Big[d_{G_1+G_2}(x)^2 + d_{G_1+G_2}(y)^2\Big] \\ &= S_1 + 2S_2 + S_3, \end{aligned}$$

where S_1, S_2, S_3 are terms of the above sums taken in order. Next we calculate S_1, S_2 and S_3 separately.

$$\begin{aligned}
S_1 &= \sum_{x\in V(G_1)} \sum_{y\in V(G_1)} d_{G_1+G_2}(x,y)\Big[d_{G_1+G_2}(x)^2 + d_{G_1+G_2}(y)^2\Big] \\
&= \sum_{x,y\in V(G_1)} d_{G_1+G_2}(x,y)\Big[d_{G_1+G_2}(x)^2 + d_{G_1+G_2}(y)^2\Big] \\
&= \sum_{xy\in G_1} d_{G_1+G_2}(x,y)\Big[d_{G_1+G_2}(x)^2 + d_{G_1+G_2}(y)^2\Big] \\
&+ \sum_{xy\notin G_1, x\neq y} d_{G_1+G_2}(x,y)\Big[d_{G_1+G_2}(x)^2 + d_{G_1+G_2}(y)^2\Big] \\
&+ \sum_{xy\notin G_1, x=y} d_{G_1+G_2}(x,y)\Big[d_{G_1+G_2}(x)^2 + d_{G_1+G_2}(y)^2\Big] \\
&= 1. \sum_{xy\in G_1} \Big[d_{G_1+G_2}(x)^2 + d_{G_1+G_2}(y)^2\Big] + 2. \sum_{xy\notin G_1, x\neq y} \Big[d_{G_1+G_2}(x)^2 + d_{G_1+G_2}(y)^2\Big] \\
&+ 0. \sum_{xy\notin G_1, x=y} \Big[d_{G_1+G_2}(x)^2 + d_{G_1+G_2}(y)^2\Big] \\
S_1 &= S_{1,1} + 2S_{1,2},
\end{aligned}$$

where $S_{1,1}$ and $S_{1,2}$ are terms of the above sums taken in order, which are computed as follows:

$$\begin{aligned}
S_{1,1} &= \sum_{xy\in G_1} \Big[d_{G_1+G_2}(x)^2 + d_{G_1+G_2}(y)^2\Big] \\
&= \sum_{xy\in G_1} \Big[(d_{G_1}(x) + n_2)^2 + (d_{G_1}(y) + n_2)^2\Big]
\end{aligned}$$

$$\begin{aligned}
&= \sum_{xy\in G_1} \Big[d_{G_1}(x)^2 + n_2^2 + 2d_{G_1}(x)n_2 + d_{G_1}(y)^2 + n_2^2 + 2n_2 d_{G_1}(y)\Big] \\
&= \sum_{xy\in G_1} \Big[d_{G_1}(x)^2 + d_{G_1}(y)^2\Big] + 2n_2 \sum_{xy\in G_1} \Big[d_{G_1}(x) + d_{G_1}(y)\Big] + 2n_2^2 \sum_{xy\in G_1} 1 \\
&= 2F(G_1) + 4n_2 M_1(G_1) + 4n_2^2 m_1 \\
S_{1,2} &= \sum_{xy\notin G_1),\ x\neq y} \Big[d_{G_1+G_2}(x)^2 + d_{G_1+G_2}(y)^2\Big] \\
&= \sum_{xy\notin G_1,\ x\neq y} \Big[(d_{G_1}(x) + n_2)^2 + (d_{G_1}(y) + n_2)^2\Big] \\
&= \sum_{xy\notin G_1,\ x\neq y} \Big[d_{G_1}(x)^2 + n_2^2 + 2d_{G_1}(x)n_2 + d_{G_1}(y)^2 + n_2^2 + 2n_2 d_{G_1}(y)\Big]
\end{aligned}$$

$$= \sum_{xy\notin G_1,\, x\neq y} \Big[d_{G_1}(x)^2 + d_{G_1}(y)^2\Big] + 2n_2 \sum_{xy\notin G_1, x\neq y} \Big[d_{G_1}(x) + d_{G_1}(y)\Big] + 2n_2^2 \sum_{xy\notin G_1,\, x\neq y} 1$$

$$= 2[M_1(G_1)(n_1-1) - F(G_1)] + 2n_2.2[(n_1-1)2m_1 - M_1(G_1)] + 2n_2^2[(n_1-1)n_1 - 2m_1]$$

$$S_2 = \sum_{x\in V(G_1)} \sum_{y\in V(G_2)} d_{G_1+G_2}(x,y)\Big[d_{G_1+G_2}(x)^2 + d_{G_1+G_2}(y)^2\Big]$$

$$= \sum_{x\in V(G_1)} \sum_{y\in V(G_2)} \Big[(d_{G_1}(x)+n_2)^2 + (d_{G_2}(y)+n_1)^2\Big]$$

$$= 1. \sum_{x\in V(G_1)} \sum_{y\in V(G_2)} \Big[d_{G_1}(x)^2 + n_2^2 + 2d_{G_1}(x)n_2 + d_{G_2}(y)^2 + n_1^2 + 2n_1 d_{G_2}(y)\Big]$$

$$= \sum_{x\in V(G_1)} d_{G_1}(x)^2 \sum_{y\in V(G_2)} 1 + n_2^2 \sum_{x\in V(G_1)} 1 \sum_{y\in V(G_2)} 1 + 2n_2 \sum_{x\in V(G_1)} d_{G_1}(x) \sum_{y\in V(G_2)} 1$$

$$+ \sum_{x\in V(G_1)} 1 \sum_{y\in V(G_2)} d_{G_2}(y)^2 + n_1^2 \sum_{x\in V(G_1)} 1 \sum_{y\in V(G_2)} 1 + 2n_1 \sum_{y\in V(G_2)} d_{G_2}(y) \sum_{x\in V(G_1)} 1$$

$$= M_1(G_1)n_2 + n_2^3 n_1 + 4n_2^2 m_1 + M_1(G_2)n_1 + n_1^3 n_2 + 4n_1^2 m_2$$

$$S_3 = \sum_{x\in V(G_2)} \sum_{y\in V(G_2)} d_{G_1+G_2}(x,y)\Big[d_{G_1+G_2}(x)^2 + d_{G_1+G_2}(y)^2\Big]$$

$$= \sum_{x,y\in V(G_2)} d_{G_1+G_2}(x,y)\Big[d_{G_1+G_2}(x)^2 + d_{G_1+G_2}(y)^2\Big]$$

$$= \sum_{xy\in G_2,} d_{G_1+G_2}(x,y)\Big[d_{G_1+G_2}(x)^2 + d_{G_1+G_2}(y)^2\Big]$$

$$+ \sum_{xy\notin G_2,\, x\neq y} d_{G_1+G_2}(x,y)\Big[d_{G_1+G_2}(x)^2 + d_{G_1+G_2}(y)^2\Big]$$

$$+ \sum_{xy\notin G_2,\, x=y} d_{G_1+G_2}(x,y)\Big[d_{G_1+G_2}(x)^2 + d_{G_1+G_2}(y)^2\Big]$$

$$= 1. \sum_{xy\in G_2} \Big[d_{G_1+G_2}(x)^2 + d_{G_1+G_2}(y)^2\Big] + 2. \sum_{xy\notin G_2,\, x\neq y} \Big[d_{G_1+G_2}(x)^2 + d_{G_1+G_2}(y)^2\Big]$$

$$+ 0. \sum_{xy\notin G_2,\, x=y} \Big[d_{G_1+G_2}(x)^2 + d_{G_1+G_2}(y)^2\Big]$$

$$S_3 = S_{3,1} + 2S_{3,2},$$

where $S_{3,1}$ and $S_{3,2}$ are terms of the above sums taken in order, which are computed as follows:

$$S_{3,1} = \sum_{xy\in G_2} \Big[d_{G_1+G_2}(x)^2 + d_{G_1+G_2}(y)^2\Big]$$

$$\begin{aligned}
&= \sum_{xy\in G_2}\Big[(d_{G_2}(x)+n_1)^2+(d_{G_2}(y)+n_1)^2\Big]\\
&= \sum_{xy\in G_2}\Big[d_{G_2}(x)^2+n_1^2+2d_{G_2}(x)n_1+d_{G_2}(y)^2+n_1^2+2n_1d_{G_2}(y)\Big]\\
&= \sum_{xy\in G_2}\Big[d_{G_2}(x)^2+d_{G_2}(y)^2\Big]+2n_1\sum_{xy\in G_2}\Big[d_{G_2}(x)+d_{G_2}(y)\Big]+2n_1^2\sum_{xy\in G_2}1\\
&= 2F(G_2)+4n_1M_1(G_2)+4n_1^2m_2\\
S_{3,2} &= \sum_{xy\notin G_2,\ x\neq y}\Big[d_{G_1+G_2}(x)^2+d_{G_1+G_2}(y)^2\Big]\\
&= \sum_{xy\notin G_2,\ x\neq y}\Big[(d_{G_2}(x)+n_1)^2+(d_{G_2}(y)+n_1)^2\Big]\\
&= \sum_{xy\notin G_2,\ x\neq y}\Big[d_{G_2}(x)^2+n_1^2+2d_{G_2}(x)n_1+d_{G_2}(y)^2+n_1^2+2n_1d_{G_2}(y)\Big]\\
&= \sum_{xy\notin G_2,\ x\neq y}\Big[d_{G_2}(x)^2+d_{G_2}(y)^2\Big]+2n_1\sum_{xy\notin G_2,\ x\neq y}\Big[d_{G_2}(x)+d_{G_2}(y)\Big]+2n_1^2\sum_{xy\notin G_2,\ x\neq y}1\\
&= 2[M_1(G_2)(n_2-1)-F(G_2)]+2n_12[(n_2-1)2m_2-M_1(G_2)]+2n_1^2[(n_2-1)n_2-2m_2]
\end{aligned}$$

$$\therefore 2\times DF(G_1+G_2)=S_1+2S_2+S_3=S_{1,1}+2S_{1,2}+2S_2+S_{3,1}+2S_{3,2} \qquad (1)$$

Substituting $S_{1,1}$, $S_{1,2}$, S_2, $S_{3,1}$, and $S_{3,2}$ in (1), the desired result follows after simple calculation. ∎

2.3 Composition

In the following theorem, we compute the distance version of F-index of the composition of two graphs.

Theorem 2.13. Let $G_i, i=1,2$, be a (n_i,m_i)- graph, put $\overline{m}_i=e(\overline{G}_i)$. Then

$$\begin{aligned}
2\times DF(G_1[G_2]) &= 4n_2^2m_2DF(G_1)+4W(G_1)F(G_2)+4n_2M_1(G_2)DD(G_1)\\
&+ 4n_2^2M_1(G_1)\Big[(n_2-1)n_2-2m_2\Big]+4n_1\Big[(n_2-1)M_1(G_2)-F(G_2)\Big]\\
&+ 16n_2m_1\Big[2m_2(n_2-1)-M_1(G_2)\Big]+4n_2^2M_1(G_1)m_2+2n_1F(G_2)\\
&+ 8n_2m_1M_1(G_2)+2n_2^2DF(G_1)(2\overline{m}_2+n_2)\\
&+ 4W(G_1)\Big[(n_2-1)M_1(G_2)-F(G_2)+M_1(G_2)\Big]\\
&+ 4n_2\Big[(n_2-1)2\overline{m}_2-M_1(\overline{G}_2)+2m_2\Big]DD(G_1)
\end{aligned}$$

Proof:

$$
\begin{aligned}
2\times\Big[DF(G_1[G_2])\Big] &= \sum_{x,y\in V(G_1)}\ \sum_{u,v\in V(G_2)} d_{G_1[G_2]}((x,u),(y,v))\Big[d_{G_1[G_2]}(x,u)^2+d_{G_1[G_2]}(y,v)^2\Big]\\
&= \sum_{x,y\in V(G_1)}\Big\{\sum_{uv\in G_2}\Big[d_{G_1[G_2]}(x,u)^2+d_{G_1[G_2]}(y,v)^2\Big]\\
&+ \sum_{uv\notin G_2}\Big[d_{G_1[G_2]}(x,u)^2+d_{G_1[G_2]}(y,v)^2\Big]\Big\}\\
&= \sum_{x,y\in V(G_1)}\ \sum_{uv\in G_2}\Big[d_{G_1[G_2]}((x,u)^2+d_{G_1[G_2]}(y,v)^2\Big]\\
&+ \sum_{x,y\in V(G_1)}\ \sum_{uv\notin G_2}\Big[d_{G_1[G_2]}((x,u)^2+d_{G_1[G_2]}((y,v)^2\Big]\\
&= \sum_{x,y\in G_1,x=y}\ \sum_{uv\in G_2}\Big[d_{G_1[G_2]}((x,u)^2+d_{G_1[G_2]}((y,v)^2\Big]\\
&+ \sum_{x,y\in G_1,x\neq y}\ \sum_{uv\in G_2}\Big[d_{G_1[G_2]}((x,u)^2+d_{G_1[G_2]}((y,v)^2\Big]\\
&+ \sum_{x,y\in G_1,x=y}\ \sum_{uv\notin G_2}\Big[d_{G_1[G_2]}((x,u)^2+d_{G_1[G_2]}((y,v)^2\Big]\\
&+ \sum_{x,y\in G_1,x\neq y}\ \sum_{uv\notin G_2}\Big[d_{G_1[G_2]}((x,u)^2+d_{G_1[G_2]}((y,v)^2\Big]\\
&= J_3+J_1+J_2+J_4,
\end{aligned}
$$

where J_3, J_1, J_2, J_4 are terms of the above sums taken in order. Next we calculate J_1, J_2, J_3 and J_4 separately one by one.

$$
\begin{aligned}
J_1 &= \sum_{x,y\in G_1,x\neq y}\ \sum_{uv\in G_2} d_{G_1[G_2]}((x,u),(y,v))\Big[d_{G_1[G_2]}(x,u)^2+d_{G_1[G_2]}(y,v)^2\Big]\\
&= \sum_{x,y\in G_1,x\neq y}\ \sum_{uv\in G_2} d_{G_1}(x,y)\Big[\Big(d_{G_2}(u)+d_{G_1}(x)n_2\Big)^2+\Big(d_{G_2}(v)+d_{G_1}(y)n_2\Big)^2\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{x,y\in G_1,x\neq y}\ \sum_{uv\in G_2} d_{G_1}(x,y)\Big[n_2^2d_{G_1}(x)^2+d_{G_2}(u)^2+2n_2d_{G_1}(x)d_{G_2}(u)+n_2^2d_{G_1}^2(y)\\
&+ d_{G_2}(v)^2+2n_2d_{G_1}(y)d_{G_2}(v)\Big]\\
&= n_2^2\sum_{x,y\in G_1,x\neq y} d_{G_1}(x,y)[d_{G_1}(x)^2+d_{G_1}(y)^2]\sum_{uv\in G_2}1\\
&+ \sum_{x,y\in G_1,x\neq y} d_{G_1}(x,y)\sum_{uv\in G_2}[d_{G_2}(u)^2+d_{G_2}(v)^2]+2n_2\sum_{x,y\in G_1,x\neq y} d_{G_1}(x,y)d_{G_1}(x)\sum_{uv\in G_2}d_{G_2}(u)
\end{aligned}
$$

$$
\begin{aligned}
& + \; 2n_2 \sum_{x,y\in G_1, x\neq y} d_{G_1}(x,y) d_{G_1}(y) \sum_{uv\in G_2} d_{G_2}(v) \\
& = \; n_2^2(2m_2)(2DDF(G_1)) + 2W(G_1)2F(G_2) + 2n_2 \sum_{x,y\in G_1, x\neq y} d_{G_1}(x,y) d_{G_1}(x) \sum_{u\in V(G_2)} d^2_{G_2}(u) \\
& + \; 2n_2 \sum_{x,y\in G_1, x\neq y} d_{G_1}(x,y) d_{G_1}(y) \sum_{u\in V(G_2)} d^2_{G_2}(v) \\
& = \; n_2^2(2m_2)(2DF(G_1)) + 4W(G_1)F(G_2) + 2n_2 M_1(G_2) \sum_{x,y\in G_1, x\neq y} d_{G_1}(x,y)[d_{G_1}(x) + d_{G_1}(y)] \\
& = \; 4n_2^2 m_2 DF(G_1) + 4W(G_1)F(G_2) + 2n_2 M_1(G_2) 2DD(G_1)
\end{aligned}
$$

$$
\begin{aligned}
J_2 & = \sum_{x,y\in G_1, x=y} \sum_{uv\notin G_2} d_{G_1[G_2]}((x,u),(y,v)) \Big[d_{G_1[G_2]}(x,u)^2 + d_{G_1[G_2]}(y,v)^2 \Big] \\
& = \sum_{x,y\in G_1, x=y} \Big\{ \sum_{uv\notin G_2,\, u=v} d_{G_1[G_2]}((x,u),(y,v)) \Big[d_{G_1[G_2]}(x,u)^2 + d_{G_1[G_2]}(y,v)^2 \Big] \\
& + \sum_{uv\notin G_2,\, u\neq v} d_{G_1[G_2]}((x,u),(y,v)) \Big[d_{G_1[G_2]}(x,u)^2 + d_{G_1[G_2]}(y,v)^2 \Big] \Big\} \\
& = \sum_{x,y\in G_1, x=y} \sum_{uv\notin G_2,\, u=v} d_{G_1[G_2]}((x,u),(y,v)) \Big[d_{G_1[G_2]}(x,u)^2 + d_{G_1[G_2]}(y,v)^2 \Big] \\
& + \sum_{x,y\in G_1, x=y} \sum_{uv\notin G_2,\, u\neq v} d_{G_1[G_2]}((x,u),(y,v)) \Big[d_{G_1[G_2]}(x,u)^2 + d_{G_1[G_2]}(y,v)^2 \Big] \Big\} \\
& = 0 + \sum_{x,y\in G_1, x=y} \sum_{uv\notin G_2,\, u\neq v} d_{G_1[G_2]}((x,u),(y,v)) \Big[d_{G_1[G_2]}(x,u)^2 + d_{G_1[G_2]}(y,v)^2 \Big] \\
& = \sum_{x,y\in G_1, x=y} \sum_{uv\notin G_2,\, u\neq v} d_{G_1}(x,y) \Big[[d_{G_2}(u) + d_{G_1}(x)n_2]^2 + [d_{G_2}(v) + d_{G_1}(y)n_2]^2 \Big] \\
& = 2\Big\{ \sum_{x,y\in G_1, x=y} \sum_{uv\notin G_2,\, u\neq v} \Big[n_2^2 d_{G_1}(x)^2 + d_{G_2}(u)^2 + 2n_2 d_{G_1}(x) d_{G_2}(u) \\
& + n_2^2 d_{G_1}(y)^2 + d_{G_2}(v)^2 + 2n_2 d_{G_1}(y) d_{G_2}(v) \Big] \Big\} \\
& = 2\Big\{ n_2^2 \sum_{x,y\in G_1, x=y} \Big(d_{G_1}(x)^2 + d_{G_1}(y)^2 \Big) \sum_{uv\notin G_2,\, u\neq v} 1 \\
& + \sum_{x,y\in G_1, x=y} 1 \sum_{uv\notin G_2,\, u\neq v} \Big(d_{G_2}(u)^2 + d_{G_2}(v)^2 \Big) + 2n_2 \sum_{x,y\in G_1, x=y} d_{G_1}(x) \sum_{uv\notin G_2,\, u\neq v} d_{G_2}(u) \Big] \\
& + 2n_2 \sum_{x,y\in G_1, x=y} d_{G_1}(y) \sum_{uv\notin G_2,\, u\neq v} d_{G_2}(v) \Big\} \\
& = 2\Big[n_2^2 2M_1(G_1)[(n_2-1)n_2 - 2m_2] + 2n_1[(n_2-1)M_1(G_2) - F(G_2)]
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad 2n_2(2m_1)2[2m_2(n_2-1) - M_1(G_2)]\Big]\\
J_3 &= \sum_{x,y\in G_1, x=y} \ \sum_{uv\in G_2} d_{G_1[G_2]}\Big((x,u)(y,v)\Big)\Big[[d_{G_1[G_2]}(x,u)^2 + d_{G_1[G_2]}(y,v)^2\Big]\\
&= 1. \sum_{x,y\in G_1, x=y} \ \sum_{uv\in G_2} \Big[[d_{G_1[G_2]}(x,u)^2 + d_{G_1[G_2]}(y,v)^2\Big]\\
&= \sum_{x,y\in G_1, x=y} \ \sum_{uv\in G_2} \Big[[d_{G_2}(u) + d_{G_1}(x)n_2]^2 + [d_{G_2}(v) + d_{G_1}(y)n_2]^2\Big]\\
&= \Big\{ \sum_{x,y\in G_1, x=y} \ \sum_{uv\in G_2} \Big[n_2^2 d_{G_1}(x)^2 + d_{G_2}(u)^2 + 2n_2 d_{G_1}(x) d_{G_2}(u)\\
&+ \quad n_2^2 d_{G_1}(y)^2 + d_{G_2}(v)^2 + 2n_2 d_{G_1}(y) d_{G_2}(v)\Big]\Big\}\\
&= \Big[n_2^2 \sum_{x,y\in G_1, x=y} \Big(d_{G_1}(x)^2 + d_{G_1}(y)^2\Big) \sum_{uv\in G_2} 1 + \sum_{x,y\in G_1, x=y} 1 \sum_{uv\in G_2} \Big(d_{G_2}(u)^2 + d_{G_2}(v)^2\Big)\\
&+ \quad 2n_2 \sum_{x,y\in G_1, x=y} d_{G_1}(x) \sum_{uv\in G_2} d_{G_2}(u) + 2n_2 \sum_{x,y\in G_1, x=y} d_{G_1}(y) \sum_{uv\in G_2} d_{G_2}(v)\Big]\\
&= n_2^2 \sum_{x\in V(G_1)} \Big(d_{G_1}(x)^2 + d_{G_1}(y)^2\Big) 2m_2 + 2n_1 F(G_2) + 2n_2 \sum_{x,y\in G_1, x=y} d_{G_1}(x) \sum_{u\in V(G_2)} d_{G_2}^2(u)\\
&+ \quad 2n_2 \sum_{x,y\in G_1, x=y} d_{G_1}(y) \sum_{v\in V(G_2)} d_{G_2}^2(v)\\
&= n_2^2(2M_1(G_1))2m_2 + 2n_1 F(G_2) + 2[2n_2(2m_1)M_1(G_2)]\\
J_4 &= \sum_{x,y\in G_1, x\neq y} \ \sum_{uv\notin G_2} d_{G_1[G_2]}(x,u)(y,v)\Big[[d_{G_1[G_2]}(x,u)^2 + d_{G_1[G_2]}(y,v)^2\Big]\\
&= \sum_{x,y\in G_1, x\neq y} \ \sum_{uv\notin G_2} d_{G_1[G_2]}(x,y)\Big[[d_{G_1[G_2]}(x,u)^2 + d_{G_1[G_2]}(y,v)^2\Big]\\
&= \sum_{x,y\in G_1, x\neq y} \ \sum_{uv\notin G_2} d_{G_1[G_2]}(x,y)\Big[[d_{G_2}(u) + d_{G_1}(x)n_2]^2 + [d_{G_2}(v) + d_{G_1}(y)n_2]^2\Big]\\
&= \sum_{x,y\in G_1, x\neq y} \ \sum_{uv\notin G_2} d_{G_1}(x,y)\Big[n_2^2 d_{G_1}(x)^2 + d_{G_2}(u)^2 + 2n_2 d_{G_1}(x) d_{G_2}(u)\\
&+ \quad n_2^2 d_{G_1}(y)^2 + d_{G_2}(v)^2 + 2n_2 d_{G_1}(y) d_{G_2}(v)\Big]\\
&= n_2^2 \sum_{x,y\in G_1, x\neq y} \Big(d(x,y)[d_{G_1}(x)^2 + d_{G_1}(y)^2]\Big) \sum_{uv\notin G_2} 1\\
&+ \quad \sum_{x,y\in G_1, x\neq y} d(x,y) \sum_{uv\notin G_2} \Big(d_{G_2}(u)^2 + d_{G_2}(v)^2\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+\ 2n_2 \sum_{x,y\in G_1,x\neq y} d(x,y)d_{G_1}(x) \sum_{uv\notin G_2} d_{G_2}(u) + 2n_2 \sum_{x,y\in G_1,x\neq y} d(x,y)d_{G_1}(y) \sum_{uv\notin G_2} d_{G_2}(v)\\
&=\ n_2^2(2DF(G_1))(2\overline{m}_2+n_2)+2W(G_1)2[(n_2-1)M_1(G_2)-F(G_2)+M_1(G_2)]\\
&+\ 2n_2\Big((n_2-1)2\overline{m}_2 - M_1(\overline{G}_2)+2m_2\Big) \sum_{x,y\in G_1,x\neq y} d(x,y)[d_{G_1}(x)+d_{G_1}(y)]\\
&=\ n_2^2(2DF(G_1))(2\overline{m}_2+n_2)+2W(G_1)2[(n_2-1)M_1(G_2)-F(G_2)+M_1(G_2)]\\
&+\ 2n_2\Big((n_2-1)2\overline{m}_2 - M_1(\overline{G}_2)+2m_2\Big)2DD(G_1)
\end{aligned}
$$

By adding J_1, J_2, J_3, and J_4 the desired result follows after simple calculation. ∎

2.4 Cartesian

Lemma 2.14. Let G_1 and G_2 be two connected graphs with vertex sets $V(G_1) = \{u_1, u_2, ..., u_n\}$ and $V(G_2) = \{v_1, v_2, ..., v_m\}$ respectively. Let $w_{ij} = (u_i, v_j)$ and $w_{pq} = (u_p, v_q)$ be in $V(G_1\square G_2)$. Then $d(G_1\square G_2)(w_{ij}, w_{pq}) = d_{G_1}(u_i, u_p) + d_{G_2}(v_j, v_q)$ and $d(G_1\square G_2)(w_{ij}) = d_{G_1}(u_i) + d_{G_2}(v_j)$.

Lemma 2.15. Let G be a grah with n vertices. Let $V(G) = \{v_1, v_2, ..., v_n\}$.

$$
\begin{aligned}
(i) \sum_{j,q=0,j\neq q}^{n-1} d_G^2(v_j) &= (n-1)M_1(G)\\
(ii) \sum_{j,q=0,j\neq q}^{n-1} d_G(v_j) &= 2(n-1)e(G)
\end{aligned}
$$

Proof:

$$
\begin{aligned}
(i) \sum_{j,q=0,j\neq q}^{n-1} d_G^2(v_j) &= \sum_{j=0}^{n-1} \sum_{q=0,j\neq q}^{n-1} d_G^2(v_j)\\
&= \sum_{j=0}^{n-1} d_G^2(v_j) \sum_{q=0,j\neq q}^{n-1} 1\\
&= (n-1)M_1(G)\\
(ii) \sum_{j,q=0,j\neq q}^{n-1} d_G(v_j) &= \sum_{j=0}^{n-1} \sum_{q=0,j\neq q}^{n-1} d_G(v_j)\\
&= \sum_{j=0}^{n-1}[2e(G)-d(v_j)]\\
&= 2e(G)(n-1)
\end{aligned}
$$

∎

In the following theorem we compute the distance version of F-index of the Cartesian product of two graphs.

Theorem 2.16. Let $G_i, i = 1, 2$, be a (n_i, m_i)- graph ,put $\overline{m}_i = e(\overline{G}_i)$. Then

$$\begin{aligned} 2 \times DF(G_1 \square G_2) &= 2DF(G_1)n_2 + 4W(G_1)M_1(G_2) + 8DD(G_1)m_2 + 4M_1(G_1)W(G_2) \\ &+ 2n_1DF(G_2) + 8m_1DD(G_2) + 2n_2(n_2-1)DF(G_1) \\ &+ 4W(G_1)(n_2-1)M_1(G_2) + 8(n_2-1)m_2DD(G_1) \\ &+ 4W(G_2)(n_1-1)M_1(G_1) + 2n_1(n_1-1)DF(G_2) + 8(n_1-1)m_1DD(G_2) \end{aligned}$$

Proof: Let $G = G_1 \square G_2$. Then,

$$\begin{aligned} 2 \times DF(G) &= \sum_{w_{ij}, w_{pq} \in V(G)} d_G(w_{ij}, w_{pq})[d_G(w_{ij})^2 + d_G(w_{pq})^2] \\ &= \sum_{j=0}^{n_2-1} \sum_{i,p=0, i \neq p}^{n_1-1} d_G(w_{ij}, w_{pj})[d_G(w_{ij})^2 + d_G(w_{pj})^2] \\ &+ \sum_{i=0}^{n_1-1} \sum_{j,q=0, j \neq p}^{n_2-1} d_G(w_{ij}, w_{iq})[d_G(w_{ij})^2 + d_G(w_{iq})^2] \\ &+ \sum_{j,q=0, j \neq q}^{n_2-1} \sum_{i,p=0, i \neq p}^{n_1-1} d_G(w_{ij}, w_{pq})[d_G(w_{ij})^2 + d_G(w_{pq})^2] \\ &= A_1 + A_2 + A_3, \end{aligned}$$

where A_1, A_2, A_3 are the sums of the above terms in order. Next we calculate A_1, A_2, and A_3 separately.

$$\begin{aligned} A_1 &= \sum_{j=0}^{n_2-1} \sum_{i,p=0, i \neq p}^{n_1-1} d_G(w_{ij}, w_{pj})[d_G(w_{ij})^2 + d_G(w_{pj})^2] \\ &= \sum_{j=0}^{n_2-1} \sum_{i,p=0, i \neq p}^{n_1-1} d_{G_1}(u_i, u_p)\Big[(d_{G_1}(u_i) + d_{G_2}(v_j)]^2 + [d_{G_1}(u_p) + d_{G_2}(v_j)]^2\Big] \\ &= \sum_{j=0}^{n_2-1} \sum_{i,p=0, i \neq p}^{n_1-1} d_{G_1}(u_i, u_p)\Big[d^2_{G_1}(u_i) + d^2_{G_2}(v_j) + 2d_{G_1}(u_i)d_{G_2}(v_j) + d^2_{G_1}(u_p) + d^2_{G_2}(v_j) \\ &+ 2d_{G_1}(u_p)d_{G_1}(v_j)\Big] \\ &= \sum_{j=0}^{n_2-1} \sum_{i,p=0, i \neq p}^{n_1-1} d_{G_1}(u_i, u_p)\Big[d^2_{G_1}(u_i) + d^2_{G_1}(v_p)] + 2d^2_{G_2}(v_j) + 2[d_{G_1}(u_i) + d_{G_1}(u_p)]d_{G_2}(v_j)\Big] \end{aligned}$$

$$\begin{aligned}
&= \sum_{i,p=0,i\neq p}^{n_1-1} d_{G_1}(u_i,u_p)[d^2_{G_1}(u_i)+d^2_{G_1}(v_p)]\sum_{j=0}^{n_2-1}1 + 2\sum_{i,p=0,i\neq p}^{n_1-1} d_{G_1}(u_i,u_p)\sum_{j=0}^{n_2-1} d^2_{G_2}(v_j)\\
&+ 2\sum_{i,p=0,i\neq p}^{n_1-1} d_{G_1}(u_i,u_p)[d_{G_1}(u_i)+d_{G_1}(u_p)]\sum_{j=0}^{n_2-1} d_{G_2}(v_j)\\
&= 2DF(G_1)n_2 + 2(2W(G_1))(M_1(G_2)) + 2(2DD(G_1))(2m_2)\\
&= 2DF(G_1)n_2 + 4W(G_1)M_1(G_2) + 8DD(G_1)m_2
\end{aligned}$$

$$\begin{aligned}
A_2 &= \sum_{i=0}^{n_1-1}\sum_{j,q=0,j\neq q}^{n_2-1} d_G(w_{ij},w_{iq})[d_G(w_{ij})^2+d_G(w_{iq})^2]\\
&= \sum_{i=0}^{n_1-1}\sum_{j,q=0,j\neq q}^{n_2-1} d_{G_2}(v_j,v_q)\Big[[d_{G_1}(u_i)+d_{G_2}(v_j)]^2+[d_{G_1}(u_i)+d_{G_2}(v_q)]^2\Big]\\
&= \sum_{i=0}^{n_1-1}\sum_{j,q=0,j\neq q}^{n_2-1} d_{G_2}(v_j,v_q)\Big[d^2_{G_1}(u_i)+d^2_{G_2}(v_j)+2d_{G_1}(u_i)d_{G_2}(v_j)+d^2_{G_1}(u_i)+d^2_{G_2}(v_q)\\
&+ 2d_{G_1}(u_i)d_{G_1}(v_q)\Big]\\
&= 2\sum_{i=0}^{n_1-1} d^2_{G_1}(u_i)\sum_{j,q=0,j\neq q}^{n_2-1} d_{G_2}(v_j,v_q) + \sum_{i=0}^{n_1-1}1\sum_{j,q=0,j\neq q}^{n_2-1} d_{G_2}(v_j,v_q)[d^2_{G_2}(v_j)+d^2_{G_2}(v_q)]\\
&+ 2\sum_{i=0}^{n_1-1} d_{G_1}(u_i)\sum_{j,q=0,j\neq q}^{n_2-1} d_{G_2}(v_j,v_q)[d_{G_2}(v_j)+d_{G_2}(v_q)]\\
&= 2M_1(G_1)(2W(G_2)+n_1(2DF(G_2))+2(2m_1)(2DD(G_2))\\
&= 4M_1(G_1)W(G_2)+2n_1DF(G_2)+8m_1DD(G_2)
\end{aligned}$$

$$\begin{aligned}
A_3 &= \sum_{j,q=0,j\neq q}^{n_2-1}\sum_{i,p=0,i\neq p}^{n_1-1} d_G(w_{ij},w_{pq})[d_G(w_{ij})^2+d_G(w_{pq})^2]\\
&= \sum_{j,q=0,j\neq q}^{n_2-1}\sum_{i,p=0,i\neq p}^{n_1-1}\Big(d_{G_1}(u_i,u_p)+d_{G_2}(v_j,v_q)\Big)\Big[[d_{G_1}(u_i)+d_{G_2}(v_j)]^2\\
&+ [d_{G_1}(u_p)+d_{G_2}(v_q)]^2\Big]\\
&= \sum_{j,q=0,j\neq q}^{n_2-1}\sum_{i,p=0,i\neq p}^{n_1-1}\Big(d_{G_1}(u_i,u_p)+d_{G_2}(v_j,v_q)\Big)\Big[d^2_{G_1}(u_i)+d^2_{G_2}(v_j)+2d_{G_1}(u_i)d_{G_2}(v_j)\\
&+ d^2_{G_1}(u_p)+d^2_{G_2}(v_q)+2d_{G_1}(u_p)d_{G_2}(v_q)\Big]\\
&= \sum_{j,q=0,j\neq q}^{n_2-1}\sum_{i,p=0,i\neq p}^{n_1-1} d_{G_1}(u_i,u_p)\Big[d^2_{G_1}(u_i)+d^2_{G_2}(v_j)+2d_{G_1}(u_i)d_{G_2}(v_j)
\end{aligned}$$

$$
\begin{aligned}
&+ \quad d^2_{G_1}(u_p) + d^2_{G_2}(v_q) + 2d_{G_1}(u_p)d_{G_2}(v_q)\Big] + \sum_{j,q=0,j\neq q}^{n_2-1} \sum_{i,p=0,i\neq p}^{n_1-1} d_{G_2}(v_j,v_q)\Big[d^2_{G_1}(u_i) + d^2_{G_2}(v_j) \\
&+ \quad 2d_{G_1}(u_i)d_{G_2}(v_j) + d^2_{G_1}(u_p) + d^2_{G_2}(v_q) + 2d_{G_1}(u_p)d_{G_2}(v_q)\Big] \\
&= \quad \sum_{j,q=0,j\neq q}^{n_2-1} 1 \sum_{i,p=0,i\neq p}^{n_1-1} d_{G_1}(u_i,u_p)[d^2_{G_1}(u_i) + d^2_{G_1}(u_p)] \\
&\quad + \sum_{i,p=0,i\neq p}^{n_1-1} d_{G_1}(u_i,u_p) \sum_{j,q=0,j\neq q}^{n_2-1} [d^2_{G_2}(v_j) + d^2_{G_2}(v_q)] \\
&\quad + 2\sum_{i,p=0,i\neq p}^{n_1-1} d_{G_1}(u_i,u_p)d_{G_1}(u_i) \sum_{j,q=0,j\neq q}^{n_2-1} d_{G_2}(v_j) \\
&\quad + 2\sum_{i,p=0,i\neq p}^{n_1-1} d_{G_1}(u_i,u_p)d_{G_1}(u_p) \sum_{j,q=0,j\neq q}^{n_2-1} d_{G_2}(v_q) \\
&\quad + \sum_{j,q=0,j\neq q}^{n_2-1} d_{G_2}(v_j,v_q) \sum_{i,p=0,i\neq p}^{n_1-1} [d^2_{G_1}(u_i) + d^2_{G_1}(u_p)] \\
&\quad + \sum_{i,p=0,i\neq p}^{n_1-1} 1 \sum_{j,q=0,j\neq q}^{n_2-1} d_{G_2}(v_j,v_q)[d^2_{G_2}(v_j) + d^2_{G_2}(v_q)] \\
&\quad + 2\sum_{i,p=0,i\neq p}^{n_1-1} d_{G_1}(u_i) \sum_{j,q=0,j\neq q}^{n_2-1} d_{G_2}(v_j,v_q)d_{G_2}(v_j) \\
&\quad + 2\sum_{i,p=0,i\neq p}^{n_1-1} d_{G_1}(u_p) \sum_{j,q=0,j\neq q}^{n_2-1} d_{G_2}(v_j,v_q)d_{G_2}(v_q) \\
&\quad = n_2(n_2-1)2DF(G_1) + 2W(G_1)(2(n_2-1)M_1(G_2)) \\
&\quad + 2\sum_{i,p=0,i\neq p}^{n_1-1} d_{G_1}(u_i,u_p)[d_{G_1}(u_i) + d_{G_1}(u_p)]2(n_2-1)m_2 \\
&\quad + 2W(G_2)(2(n_1-1)M_1(G_2)) + n_1(n_1-1)2DF(G_2) \\
&\quad + 2\sum_{j,q=0,j\neq q}^{n_2-1} d_{G_2}(v_j,v_q)[d_{G_2}(v_j) + d_{G_2}(v_q)]2(n_1-1)m_1 \\
&\quad = 2n_2(n_2-1)DF(G_1) + 2W(G_1)(2(n_2-1)M_1(G_2)) + 8(n_2-1)m_2DD(G_1) \\
&\quad + 2W(G_2)(2(n_1-1)M_1(G_1)) + 2n_1(n_1-1)DF(G_2) + 8(n_1-1)m_1DD(G_2)
\end{aligned}
$$

By adding A_1, A_2 and A_3, we get the desired result. ∎

2.5 Disjunction

In the following theorem, we find the degree distance of $F-$ index of the disjunction of two graphs.

Theorem 2.17. The degree distance of $F-$ index of $G_1 \vee G_2$ is given by $2 \times DF(G_1 \vee G_2)$

$$\begin{aligned}
&= 2\Big(2n_2^2 m_2 + F(G_2) - 2n_2 M_1(G_2)\Big)\Big[(n_1-1)M_1(G_1) - F(G_1) + M_1(G_1)\Big] \\
&+ 2\Big(2n_1^2 m_1 + F(G_1) - 2n_1 M_1(G_1)\Big)\Big[(n_2-1)M_1(G_2) - F(G_2) + M_1(G_2)\Big] \\
&+ 2n_1^2(2\overline{m}_1 + n_1)F(G_2) + 4n_1\Big(n_2 M_1(G_2) - F(G_2)\Big)\Big[2\overline{m}_1(n_1-1) + 2m_1 - M_1(\overline{G}_1)\Big] \\
&+ 2n_2^2(2\overline{m}_2 + n_2)F(G_1) + 4n_2\Big(n_1 M_1(G_1) - F(G_1)\Big)\Big[2\overline{m}_2(n_2-1) + 2m_2 - M_1(\overline{G}_2)\Big] \\
&+ 4n_2^2 F(G_1)m_2 + 4n_1^2 m_1 F(G_2) + 2F(G_1)F(G_2) + 4n_1 n_2 M_1(G_1)M_1(G_2)
\end{aligned}$$

$$\begin{aligned}
&- 4n_2 F(G_1)M_1(G_2) + 4n_2^2\Big[(n_1-1)M_1(G_1) - F(G_1) + M_1(G_1)\Big](2\overline{m}_2 + n_2) \\
&- 4n_1 M_1(G_1)F(G_2) + 4n_1^2\Big[(n_2-1)M_1(G_2) - F(G_2) + M_1(G_2)\Big](2\overline{m}_1 + n_1) \\
&+ 4\Big[(n_1-1)M_1(G_1) - F(G_1) + M_1(G_1)\Big]\Big[(n_2-1)M_1(G_2) - F(G_2) + M_1(G_2)\Big] \\
&+ 8n_1 n_2\Big[2\overline{m}_1(n_1-1) + 2m_1 - M_1(\overline{G}_1))(2\overline{m}_2(n_2-1) + 2m_2 - M_1(\overline{G}_2)\Big] \\
&- 8n_1\Big(2\overline{m}_1(n_1-1) + 2m_1 - M_1(\overline{G}_1)\Big)\Big((n_2-1)M_1(G_2) - F(G_2) + M_1(G_2)\Big) \\
&- 8n_2\Big(2\overline{m}_2(n_2-1) + 2m_2 - M_1(\overline{G}_2)\Big)\Big((n_1-1)M_1(G_1) - F(G_1) + M_1(G_1)\Big) - 4n_2^3 M_1(G_1) \\
&- 4n_1^3 M_1(G_2) - 4M_1(G_1)M_1(G_2) - 32n_1 n_2 m_1 m_2 + 16n_1 m_1 M_1(G_2) + 16n_2 M_1(G_1)m_2
\end{aligned}$$

Proof: Let $G = G_1 \vee G_2$

$$\begin{aligned}
2 \times DF(G_1 \vee G_2) &= \sum_{x,y \in V(G_1)} \sum_{u,v \in V(G_2)} d_G((x,u)(y,v))[d_G(x,u)^2 + d_G(y,v)^2] \\
&= \sum_{x,y \in V(G_1)} \Big\{ \sum_{uv \in G_2} d_G((x,u)(y,v))[d_G(x,u)^2 + d_G(y,v)^2] \\
&+ \sum_{uv \notin G_2} d_G((x,u)(y,v))[d_G(x,u)^2 + d_G(y,v)^2]\Big\} \\
&= \sum_{x,y \in V(G_1)} \sum_{uv \in G_2} d_G((x,u)(y,v))[d_G(x,u)^2 + d_G(y,v)^2] \\
&+ \sum_{x,y \in V(G_1)} \sum_{uv \notin G_2} d_G((x,u)(y,v))[d_G(x,u)^2 + d_G(y,v)^2] \\
&= \sum_{xy \in G_1} \sum_{uv \in G_2} d_G((x,u)(y,v))[d_G(x,u)^2 + d_G(y,v)^2] \\
&+ \sum_{xy \notin G_1} \sum_{uv \in G_2} d_G((x,u)(y,v))[d_G(x,u)^2 + d_G(y,v)^2] \\
&+ \sum_{xy \in G_1} \sum_{uv \notin G_2} d_G((x,u)(y,v))[d_G(x,u)^2 + d_G(y,v)^2]
\end{aligned}$$

$$\begin{aligned} &+ \sum_{xy\notin G_1}\sum_{uv\notin G_2} d_G((x,u)(y,v))[d_G(x,u)^2+d_G(y,v)^2] \\ &= C_3+C_1+C_2+C_4, \end{aligned}$$

where $C_3,\ C_1,\ C_2$ and C_4 are terms of the above sums taken in order.

We compute $C_1,\ C_2,\ C_3$ and C_4 separately as follows:

$$\begin{aligned} C_1 &= \sum_{xy\notin G_1}\sum_{uv\in G_2} d_G((x,u)(y,v))[d_G(x,u)^2+d_G(y,v)^2] \\ &= \sum_{xy\notin G_1}\sum_{uv\in G_2} [d_G(x,u)^2+d_G(y,v)^2] \end{aligned}$$

$$\begin{aligned} &= \sum_{xy\notin G_1}\sum_{uv\in G_2}\Big[\Big(n_2d_{G_1}(x)+n_1d_{G_2}(u)-d_{G_1}(x)d_{G_2}(u)\Big)^2+\Big(n_2d_{G_1}(y)+n_1d_{G_2}(v) \\ &- d_{G_1}(y)d_{G_2}(v)\Big)^2\Big] \\ &= \sum_{xy\in G_1}\sum_{uv\notin G_2}\Big[n_2^2d_{G_1}^2(x)+n_1^2d_{G_2}^2(u)+d_{G_1}^2(x)d_{G_2}^2(u)+2n_1n_2d_{G_1}(x)d_{G_2}(u) \\ &- 2n_1d_{G_1}(x)d_{G_2}^2(u)-2n_2d_{G_1}^2(x)d_{G_2}(u)+n_2^2d_{G_1}^2(y)+n_1^2d_{G_2}^2(v)+d_{G_1}^2(y)d_{G_2}^2(v) \\ &+ 2n_1n_2d_{G_1}(y)d_{G_2}(v)-2n_1d_{G_1}(y)d_{G_2}^2(v)-2n_2d_{G_1}^2(y)d_{G_2}(v)\Big] \\ &= n_2^2\sum_{xy\notin G_1}\sum_{uv\in G_2}\Big(d_{G_1}^2(x)+d_{G_1}^2(y)\Big)+n_1^2\sum_{xy\notin G_1}\sum_{uv\in G_2}[d_{G_2}^2(u)+d_{G_2}^2(v)] \\ &+ \sum_{xy\notin G_1}\sum_{uv\in G_2}[d_{G_1}^2(x)d_{G_2}^2(u)+d_{G_1}^2(y)d_{G_2}^2(v)] \\ &+ 2n_1n_2\sum_{xy\notin G_1}\sum_{uv\in G_2}[d_{G_1}(x)d_{G_2}(u)+d_{G_1}(y)d_{G_2}(v)] \\ &- 2n_1\sum_{xy\notin G_1}\sum_{uv\in G_2}[d_{G_1}(x)d_{G_2}^2(u)+d_{G_2}^2(v)d_{G_1}(y)] \\ &- 2n_2\sum_{xy\notin G_1}\sum_{uv\in G_2}[d_{G_1}^2(x)d_{G_2}(u)+d_{G_1}^2(y)d_{G_2}(v)] \\ &= n_2^2\sum_{xy\notin G_1}\Big(d_{G_1}^2(x)+d_{G_1}^2(y)\Big)\sum_{uv\in G_2}1+n_1^2\sum_{xy\notin G_1}1\sum_{uv\in G_2}[d_{G_2}^2(u)+d_{G_2}^2(v)] \\ &+ \sum_{xy\notin G_1}[d_{G_1}^2(x)+d_{G_1}^2(y)]\sum_{u\in G_2}d_{G_2}^3(u)+2n_1n_2\sum_{xy\notin G_1}[d_{G_1}(x)+d_{G_1}(y)]\sum_{u\in V(G_1)}d_{G_2}^2(u) \\ &- 2n_1\sum_{xy\notin G_1}[d_{G_1}(x)+d_{G_1}(y)]\sum_{u\in V(G_1)}d_{G_2}^3(u)-2n_2\sum_{xy\notin G_1}[d_{G_1}^2(x)+d_{G_1}^2(y)]\sum_{u\in V(G_1)}d_{G_2}^2(u) \\ &= n_2^2(2[(n_1-1)M_1(G_1)-F(G_1)+M_1(G_1)])(2m_2)+n_1^2(2\overline{m}_1+n_1)2F(G_2) \\ &+ 2[(n_1-1)M_1(G_1)-F(G_1)+M_1(G_1)]F(G_2)+2n_1n_2\Big(2[2\overline{m}_1(n_1-1) \end{aligned}$$

$$
\begin{aligned}
&+\ 2m_1 - M_1(\overline{G}_1)]\Big)M_1(G_2) - 2n_1(2[2\overline{m}_1(n_1-1) + 2m_1 - M_1(\overline{G}_1)])F(G_2)\\
&-\ 2n_2(2[(n_1-1)M_1(G_1) - F(G_1) + M_1(G_1)])(M_1(G_2))\\
C_2 &= \sum_{xy\in G_1}\sum_{uv\notin G_2} d_G((x,u)(y,v))[d_G(x,u)^2 + d_G(y,v)^2]\\
&= \sum_{xy\in G_1}\sum_{uv\notin G_2}[d_G(x,u)^2 + d_G(y,v)^2]\\
&= \sum_{xy\in G_1}\sum_{uv\notin G_2}\Big[\Big(n_2d_{G_1}(x) + n_1d_{G_2}(u) - d_{G_1}(x)d_{G_2}(u)\Big)^2 + \Big(n_2d_{G_1}(y) + n_1d_{G_2}(v)\\
&-\ d_{G_1}(y)d_{G_2}(v)\Big)^2\Big]\\
&= \sum_{xy\in G_1}\sum_{uv\notin G_2}\Big[n_2^2d_{G_1}^2(x) + n_1^2d_{G_2}^2(u) + d_{G_1}^2(x)d_{G_2}^2(u) + 2n_1n_2d_{G_1}(x)d_{G_2}(u)\\
&-\ 2n_1d_{G_1}(x)d_{G_2}^2(u) - 2n_2d_{G_1}^2(x)d_{G_2}(u) + n_2^2d_{G_1}^2(y) + n_1^2d_{G_2}^2(v) + d_{G_1}^2(y)d_{G_2}^2(v)\\
&+\ 2n_1n_2d_{G_1}(y)d_{G_2}(v) - 2n_1d_{G_1}(y)d_{G_2}^2(v) - 2n_2d_{G_1}^2(y)d_{G_2}(v)\Big]\\
&= n_2^2\sum_{xy\in G_1}\sum_{uv\notin G_2}\Big(d_{G_1}^2(x) + d_{G_1}^2(y)\Big) + n_1^2\sum_{xy\in G_1}\sum_{uv\notin G_2}[d_{G_2}^2(u) + d_{G_2}^2(v)]\\
&+\ \sum_{xy\in G_1}\sum_{uv\notin G_2}[d_{G_1}^2(x)d_{G_2}^2(u) + d_{G_1}^2(y)d_{G_2}^2(v)]\\
&+\ 2n_1n_2\sum_{xy\in G_1}\sum_{uv\notin G_2}[d_{G_1}(x)d_{G_2}(u) + d_{G_1}(y)d_{G_2}(v)]\\
&-\ 2n_1\sum_{xy\in G_1}\sum_{uv\notin G_2}[d_{G_1}(x)d_{G_2}^2(u) + d_{G_2}^2(v)d_{G_1}(y)]\\
&-\ 2n_2\sum_{xy\in G_1)}\sum_{uv\notin G_2}[d_{G_1}^2(x)d_{G_2}(u) + d_{G_1}^2(y)d_{G_2}(v)]\\
&= n_2^2\sum_{xy\in G_1}\Big(d_{G_1}^2(x) + d_{G_1}^2(y)\Big)\sum_{uv\notin G_2}1 + n_1^2\sum_{xy\in G_1}1\sum_{uv\notin G_2}[d_{G_2}^2(u) + d_{G_2}^2(v)]\\
&+\ \sum_{xy\in G_1}d_{G_1}^2(x)\sum_{uv\notin G_2}[d_{G_2}^2(u) + d_{G_2}^2(v)] + 2n_1n_2\sum_{xy\in G_1}d_{G_1}(x)\sum_{uv\notin G_2}[d_{G_2}(u) + d_{G_2}(v)]\\
&-\ 2n_1\sum_{xy\in G_1}d_{G_1}(x)\sum_{uv\notin G_2}[d_{G_2}^2(u) + d_{G_2}^2(v)] - 2n_2\sum_{xy\in G_1}d_{G_1}^2(x)\sum_{uv\notin G_2}[d_{G_2}(u) + d_{G_2}(v)]\\
&= n_2^2\sum_{xy\in G_1}\Big(d_{G_1}^2(x) + d_{G_1}^2(y)\Big)\Big(\sum_{uv\notin G_2}1\Big)\\
&+\ n_1^2\sum_{xy\in G_1}1\sum_{uv\notin G_2}[d_{G_2}^2(u) + d_{G_2}^2(v)] + \sum_{x\in V(G_1)}d_{G_1}^3(x)\sum_{uv\notin G_2}[d_{G_2}^2(u) + d_{G_2}^2(v)]\\
&+\ 2n_1n_2\sum_{x\in V(G_1)}d_{G_1}^2(x)\sum_{uv\notin G_2}[d_{G_2}(u) + d_{G_2}(v)] - 2n_1\sum_{x\in V(G_1)}d_{G_1}^2(x)\sum_{uv\notin G_2}[d_{G_2}^2(u) + d_{G_2}^2(v)]
\end{aligned}
$$

$$
\begin{aligned}
&- \; 2n_2 \sum_{x\in V(G_1)} d^3_{G_1}(x) \sum_{uv\notin G_2} [d_{G_2}(u) + d_{G_2}(v)] \\
&= \; n_2^2(2F(G_1))(2\overline{m}_2 + n_2) + n_1^2(2m_1)\Big(2[(n_2-1)M_1(G_2) - F(G_2) + M_1(G_2)]\Big) \\
&+ \; 2[(n_2-1)M_1(G_2) - F(G_2) + M_1(G_2)]F(G_1) \\
&+ \; 2n_1n_2M_1(G_1)2\Big[2\overline{m}_2(n_2-1) + 2m_2 - M_1(\overline{G_2})\Big] \\
&- \; 2n_1M_1(G_1)\Big(2[(n_2-1)M_1(G_2) - F(G_2) + M_1(G_2)]\Big) \\
&- \; 2n_2F(G_1)\Big(2[2\overline{m}_2(n_2-1) + 2m_2 - M_1(\overline{G}_2)]\Big)
\end{aligned}
$$

$$
\begin{aligned}
C_3 &= \sum_{xy\in G_1)} \sum_{uv\in G_2)} d_G((x,u)(y,v))[d_G(x,u)^2 + d_G(y,v)^2] \\
&= \sum_{xy\in G_1} \sum_{uv\in G_2} [d_G(x,u)^2 + d_G(y,v)^2] \\
&= \sum_{xy\in G_1} \sum_{uv\in G_2} \Big[\Big(n_2d_{G_1}(x) + n_1d_{G_2}(u) - d_{G_1}(x)d_{G_2}(u)\Big)^2 + \Big(n_2d_{G_1}(y) + n_1d_{G_2}(v) \\
&- \; d_{G_1}(y)d_{G_2}(v)\Big)^2\Big] \\
&= \sum_{xy\in G_1} \sum_{uv\in G_2} \Big[n_2^2d^2_{G_1}(x) + n_1^2d^2_{G_2}(u) + d^2_{G_1}(x)d^2_{G_2}(u) + 2n_1n_2d_{G_1}(x)d_{G_2}(u) \\
&- \; 2n_1d_{G_1}(x)d^2_{G_2}(u) - 2n_2d^2_{G_1}(x)d_{G_2}(u) + n_2^2d^2_{G_1}(y) + n_1^2d^2_{G_2}(v) + d^2_{G_1}(y)d^2_{G_2}(v) \\
&+ \; 2n_1n_2d_{G_1}(y)d_{G_2}(v) - 2n_1d_{G_1}(y)d^2_{G_2}(v) - 2n_2d^2_{G_1}(y)d_{G_2}(v)\Big] \\
&= \; n_2^2 \sum_{xy\in E(G_1)} \sum_{uv\in E(G_2)} \Big(d^2_{G_1}(x) + d^2_{G_1}(y)\Big) + n_1^2 \sum_{xy\in E(G_1)} \sum_{uv\in E(G_2)} [d^2_{G_2}(u) + d^2_{G_2}(v)] \\
&+ \sum_{xy\in E(G_1)} \sum_{uv\in E(G_2)} [d^2_{G_1}(x)d^2_{G_2}(u) + d^2_{G_1}(y)d^2_{G_2}(v)] \\
&+ \; 2n_1n_2 \sum_{xy\in E(G_1)} \sum_{uv\in E(G_2)} [d_{G_1}(x)d_{G_2}(u) + d_{G_1}(y)d_{G_2}(v)] \\
&- \; 2n_1 \sum_{xy\in E(G_1)} \sum_{uv\in E(G_2)} [d_{G_1}(x)d^2_{G_2}(u) + d^2_{G_2}(v)d_{G_1}(y)] \\
&- \; 2n_2 \sum_{xy\in E(G_1)} \sum_{uv\in E(G_2)} [d^2_{G_1}(x)d_{G_2}(u) + d^2_{G_1}(y)d_{G_2}(v)] \\
&= \; n_2^2 \sum_{xy\in E(G_1)} \Big(d^2_{G_1}(x) + d^2_{G_1}(y)\Big) \sum_{uv\in E(G_2)} 1 + n_1^2 \sum_{xy\in G_1} 1 \sum_{uv\in G_2} [d^2_{G_2}(u) + d^2_{G_2}(v)] \\
&+ \sum_{xy\in G_1} d^2_{G_1}(x) \sum_{uv\in G_2} [d^2_{G_2}(u) + d^2_{G_2}(v)] + 2n_1n_2 \sum_{xy\in G_1} [d_{G_1}(x) + d_{G_1}(y)] \sum_{u\in G_2} d^2_{G_2}(u) \\
&- \; 2n_1 \sum_{xy\in G_1} [d_{G_1}(x) + d_{G_1}(y)] \sum_{u\in G_2} d^3_{G_2}(u) - 2n_2 \sum_{xy\in G_1} [d^2_{G_1}(x) + d^2_{G_1}(y)] \sum_{u\in G_2} d^2_{G_2}(u)
\end{aligned}
$$

$$
\begin{aligned}
&= n_2^2(2F(G_1))2m_2 + n_1^2(2m_1)(2F(G_2)) + 2F(G_1)F(G_2) + 2n_1n_2(2M_1(G_1))M_1(G_2) \\
&- 2n_1(2M_1(G_1))(F(G_2)) - 2n_2(2F(G_1))(M_1(G_2)) \\
C_4 &= \sum_{xy\notin G_1} \sum_{uv\notin G_2} d_G((x,u)(y,v))[d_G(x,u)^2 + d_G(y,v)^2] \\
&= \sum_{xy\notin G_1} \left\{ \sum_{uv\notin G_2, u\neq v} d_G((x,u)(y,v))[d_G(x,u)^2 + d_G(y,v)^2] \right. \\
&+ \left. \sum_{uv\notin G_2, u=v} d_G((x,u)(y,v)) d_G((x,u)(y,v))[d_G(x,u)^2 + d_G(y,v)^2] \right\} \\
&= \sum_{xy\notin G_1} \sum_{uv\notin G_2, u\neq v} d_G((x,u)(y,v))[d_G(x,u)^2 + d_G(y,v)^2] \\
&+ \sum_{xy\notin G_1} \sum_{uv\notin G_2, u=v} d_G((x,u)(y,v))[d_G(x,u)^2 + d_G(y,v)^2] \\
&= \sum_{xy\notin G_1, x\neq y} \sum_{uv\notin G_2, u\neq v} d_G((x,u)(y,v))[d_G(x,u)^2 + d_G(y,v)^2] \\
&+ \sum_{xy\notin G_1, x=y} \sum_{uv\notin G_2, u\neq v} d_G((x,u)(y,v))[d_G(x,u)^2 + d_G(y,v)^2] \\
&+ \sum_{xy\notin G_1, x\neq y} \sum_{uv\notin G_2, u=v} d_G((x,u)(y,v))[d_G(x,u)^2 + d_G(y,v)^2] \\
&+ \sum_{xy\notin G_1, x=y} \sum_{uv\notin G_2, u=v} d_G((x,u)(y,v))[d_G(x,u)^2 + d_G(y,v)^2]
\end{aligned}
$$

$$
\begin{aligned}
&= 2 \sum_{xy\notin G_1, x\neq y} \sum_{uv\notin G_2, u\neq v} [d_G(x,u)^2 + d_G(y,v)^2] \\
&+ 2 \sum_{xy\notin G_1, x=y} \sum_{uv\notin G_2, u\neq v} [d_G(x,u)^2 + d_G(y,v)^2] \\
&+ 2 \sum_{xy\notin G_1, x\neq y} \sum_{uv\notin G_2, u=v} [d_G(x,u)^2 + d_G(y,v)^2] + 0 \\
&= 2 \sum_{xy\notin G_1, x\neq y} \sum_{uv\notin G_2, u\neq v} [d_G(x,u)^2 + d_G(y,v)^2] \\
&+ 2 \sum_{xy\notin G_1, x=y} \sum_{uv\notin G_2, u\neq v} [d_G(x,u)^2 + d_G(y,v)^2] \\
&+ 2 \sum_{xy\notin G_1, x\neq y} \sum_{uv\notin G_2, u=v} [d_G(x,u)^2 + d_G(y,v)^2] \\
&+ 2 \sum_{xy\notin G_1, x=y} \sum_{uv\notin G_2, u=v} [d_G(x,u)^2 + d_G(y,v)^2] \\
&- 2 \sum_{xy\notin G_1, x=y} \sum_{uv\notin G_2, u=v} [d_G(x,u)^2 + d_G(y,v)^2] \\
&= 2 \sum_{xy\notin G_1} \sum_{uv\notin G_2} [d_G(x,u)^2 + d_G(y,v)^2]
\end{aligned}
$$

$$
\begin{aligned}
&- \;2\sum_{xy\notin G_1, x=y}\;\sum_{uv\notin G_2, u=v}[d_G(x,u)^2+d_G(y,v)^2]\\
&= \;2C_5-2C_6, \text{where } C_5 \text{ and } C_6 \text{ are terms of the above sums taken in order.}
\end{aligned}
$$

$$
\begin{aligned}
C_5 &= \sum_{xy\notin G_1}\;\sum_{uv\notin G_2}[d_G(x,u)^2+d_G(y,v)^2]\\
&= \sum_{xy\notin G_1}\;\sum_{uv\notin (G_2}\Big[\Big(n_2d_{G_1}(x)+n_1d_{G_2}(u)-d_{G_1}(x)d_{G_2}(u)\Big)^2+\Big(n_2d_{G_1}(y)+n_1d_{G_2}(v)\\
&- \;d_{G_1}(y)d_{G_2}(v)\Big)^2\Big]\\
&= \sum_{xy\notin G_1}\;\sum_{uv\notin G_2}\Big[n_2^2d_{G_1}^2(x)+n_1^2d_{G_2}^2(u)+d_{G_1}^2(x)d_{G_2}^2(u)+2n_1n_2d_{G_1}(x)d_{G_2}(u)\\
&- \;2n_1d_{G_1}(x)d_{G_2}^2(u)-2n_2d_{G_1}^2(x)d_{G_2}(u)+n_2^2d_{G_1}^2(y)+n_1^2d_{G_2}^2(v)+d_{G_1}^2(y)d_{G_2}^2(v)\\
&+ \;2n_1n_2d_{G_1}(y)d_{G_2}(v)-2n_1d_{G_1}(y)d_{G_2}^2(v)-2n_2d_{G_1}^2(y)d_{G_2}(v)\Big]\\
&= \;n_2^2\sum_{xy\notin G_1}\;\sum_{uv\notin G_2}\Big(d_{G_1}^2(x)+d_{G_1}^2(y)\Big)+n_1^2\sum_{xy\notin G_1}\;\sum_{uv\notin G_2}[d_{G_2}^2(u)+d_{G_2}^2(v)]\\
&+ \sum_{xy\notin G_1}\;\sum_{uv\notin G_2)}[d_{G_1}^2(x)d_{G_2}^2(u)+d_{G_1}^2(y)d_{G_2}^2(v)]\\
&+ \;2n_1n_2\sum_{xy\notin G_1}\;\sum_{uv\notin G_2}[d_{G_1}(x)d_{G_2}(u)+d_{G_1}(y)d_{G_2}(v)]
\end{aligned}
$$

$$
\begin{aligned}
&- \;2n_1\sum_{xy\notin G_1}\;\sum_{uv\notin G_2}[d_{G_1}(x)d_{G_2}^2(u)+d_{G_2}^2(v)d_{G_1}(y)]\\
&- \;2n_2\sum_{xy\notin G_1}\;\sum_{uv\notin G_2}[d_{G_1}^2(x)d_{G_2}(u)+d_{G_1}^2(y)d_{G_2}(v)]\\
&= \;n_2^2\sum_{xy\notin G_1}\Big(d_{G_1}^2(x)+d_{G_1}^2(y)\Big)\sum_{uv\notin G_2}1+n_1^2\sum_{xy\notin G_1}1\;\sum_{uv\notin G_2}[d_{G_2}^2(u)+d_{G_2}^2(v)]\\
&+ \sum_{xy\notin G_1}d_{G_1}^2(x)\sum_{uv\notin G_2)}d_{G_2}^2(u)+\sum_{xy\notin G_1}d_{G_1}^2(y)\sum_{uv\notin G_2)}d_{G_2}^2(v)\\
&+ \;2n_1n_2\Big[\sum_{xy\notin G_1}d_{G_1}(x)\sum_{uv\notin G_2}d_{G_2}(u)+\sum_{xy\notin G_1}d_{G_1}(y)\sum_{uv\notin G_2}d_{G_2}(v)\Big]\\
&- \;2n_1\Big[\sum_{xy\notin G_1}d_{G_1}(x)\sum_{uv\notin G_2}d_{G_2}^2(u)+\sum_{uv\notin G_2}d_{G_2}^2(v)\sum_{xy\notin G_1}d_{G_1}(y)\Big]\\
&- \;2n_2\Big[\sum_{xy\notin G_1}d_{G_1}^2(x)\sum_{uv\notin G_2}d_{G_2}(u)+\sum_{xy\notin G_1}d_{G_1}^2(y)\sum_{uv\notin G_2}d_{G_2}(v)\Big]\\
&= \;n_2^2\,2\Big[(n_1-1)M_1(G_1)-F(G_1)+M_1(G_1)\Big](2\overline{m}_2+n_2)
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad n_1^2\, 2\Big[(n_2-1)M_1(G_2) - F(G_2) + M_1(G_2)\Big](2\overline{m}_1 + n_1)\\
&+ \quad 2\Big[(n_1-1)M_1(G_1) - F(G_1) + M_1(G_1)\Big]\Big[(n_2-1)M_1(G_2) - F(G_2) + M_1(G_2)\Big]\\
&+ \quad 2n_1n_2\Big[2(2\overline{m}_1(n_1-1) + 2m_1 - M_1(\overline{G}_1))(2\overline{m}_2(n_2-1) + 2m_2 - M_1(\overline{G}_2))\Big]\\
&- \quad 2n_1\Big[2(2\overline{m}_1(n_1-1) + 2m_1 - M_1(\overline{G}_1))((n_2-1)M_1(G_2) - F(G_2) + M_1(G_2))\Big]\\
&- \quad 2n_2\Big[2(2\overline{m}_2(n_2-1) + 2m_2 - M_1(\overline{G}_2))((n_1-1)M_1(G_1) - F(G_1) + M_1(G_1))\Big]
\end{aligned}
$$

$$
\begin{aligned}
C_6 &= \sum_{xy\notin G_1,\, x=y} \quad \sum_{uv\notin G_2,\, u=v} [d_G(x,u)^2 + d_G(y,v)^2]\\
&= \sum_{xy\notin G_1,\, x=y} \quad \sum_{uv\notin (G_2,\, u=v} \Big[\Big(n_2 d_{G_1}(x) + n_1 d_{G_2}(u) - d_{G_1}(x)d_{G_2}(u)\Big)^2 + \Big(n_2 d_{G_1}(y) + n_1 d_{G_2}(v)\\
&- \quad d_{G_1}(y)d_{G_2}(v)\Big)^2\Big]\\
&= \sum_{xy\notin G_1,\, x=y} \quad \sum_{uv\notin G_2,\, u=v} \Big[n_2^2 d_{G_1}^2(x) + n_1^2 d_{G_2}^2(u) + d_{G_1}^2(x)d_{G_2}^2(u) + 2n_1n_2 d_{G_1}(x)d_{G_2}(u)\\
&- \quad 2n_1 d_{G_1}(x)d_{G_2}^2(u) - 2n_2 d_{G_1}^2(x)d_{G_2}(u) + n_2^2 d_{G_1}^2(y) + n_1^2 d_{G_2}^2(v) + d_{G_1}^2(y)d_{G_2}^2(v)\\
&+ \quad 2n_1n_2 d_{G_1}(y)d_{G_2}(v) - 2n_1 d_{G_1}(y)d_{G_2}^2(v) - 2n_2 d_{G_1}^2(y)d_{G_2}(v)\Big]\\
&= n_2^2 \sum_{xy\notin G_1,\, x=y} \quad \sum_{uv\notin G_2,\, u=v} \Big(d_{G_1}^2(x) + d_{G_1}^2(y)\Big) + n_1^2 \sum_{xy\notin G_1,\, x=y} \quad \sum_{uv\notin G_2,\, u=v} [d_{G_2}^2(u) + d_{G_2}^2(v)]\\
&+ \sum_{xy\notin G_1,\, x=y} \quad \sum_{uv\notin G_2,\, u=v} [d_{G_1}^2(x)d_{G_2}^2(u) + d_{G_1}^2(y)d_{G_2}^2(v)]
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad 2n_1n_2 \sum_{xy\notin G_1,\, x=y} \quad \sum_{uv\notin G_2,\, u=v} [d_{G_1}(x)d_{G_2}(u) + d_{G_1}(y)d_{G_2}(v)]\\
&- \quad 2n_1 \sum_{xy\notin G_1,\, x=y} \quad \sum_{uv\notin G_2,\, u=v} [d_{G_1}(x)d_{G_2}^2(u) + d_{G_2}^2(v)d_{G_1}(y)]\\
&- \quad 2n_2 \sum_{xy\notin G_1,\, x=y} \quad \sum_{uv\notin G_2,\, u=v} [d_{G_1}^2(x)d_{G_2}(u) + d_{G_1}^2(y)d_{G_2}(v)]\\
&= \quad n_2^2 \sum_{xy\notin G_1,\, x=y} \Big(d_{G_1}^2(x) + d_{G_1}^2(y)\Big) \sum_{uv\notin G_2,\, u=v} 1 + n_1^2 \sum_{xy\notin G_1,\, x=y} 1 \quad \sum_{uv\notin G_2,\, u=v} [d_{G_2}^2(u) + d_{G_2}^2(v)]\\
&+ \quad \sum_{xy\notin G_1,\, x=y} d_{G_1}^2(x) \quad \sum_{uv\notin G_2,\, u=v} d_{G_2}^2(u) + \sum_{xy\notin G_1,\, x=y} d_{G_1}^2(y) \quad \sum_{uv\notin G_2,,\, u=v} d_{G_2}^2(v)\\
&+ \quad 2n_1n_2\Big[\sum_{xy\notin G_1,\, x=y} d_{G_1}(x) \sum_{uv\notin G_2,\, u=v} d_{G_2}(u) + \sum_{xy\notin G_1,\, x=y} d_{G_1}(y) \sum_{uv\notin G_2,\, u=v} d_{G_2}(v)\Big]
\end{aligned}
$$

$$
\begin{aligned}
- & \; 2n_1\Big[\sum_{xy\notin G_1,\, x=y} d_{G_1}(x) \sum_{uv\notin G_2,\, u=v} d^2_{G_2}(u) + \sum_{uv\notin G_2,\, u=v} d^2_{G_2}(v) \sum_{xy\notin G_1,\, x=y} d_{G_1}(y)\Big] \\
- & \; 2n_2\Big[\sum_{xy\notin G_1,\, x=y} d^2_{G_1}(x) \sum_{uv\notin G_2,\, u=v} d_{G_2}(u) + \sum_{xy\notin G_1,\, x=y} d^2_{G_1}(y) \sum_{uv\notin G_2,\, u=v} d_{G_2}(v)\Big] \\
= & \; n_2^2(2M_1(G_1))n_2 + n_1^2(2M_1(G_2))n_1 + 2M_1(G_1)M_1(G_2) + 2n_1n_2\, 2(2m_1)(2m_2) \\
- & \; 2n_1\; 2(2m_1)M_1(G_2) - 2n_2\; 2M_1(G_1)(2m_2)
\end{aligned}
$$

$$\therefore 2 \times DF(G_1 \vee G_2) = C_3 + C_1 + C_2 + C_4 = C_3 + C_1 + C_2 + 2C_5 - 2C_6 \qquad (2)$$

Substituting $C_3,\ C_1,\ C_2,\ C_5$ and C_6 in (2), the desired result follows after simple calculation. ∎

2.6 Symmetric difference

In the following theorem, we obtain the distance version of F- index of the symmetric difference of two graphs.

Theorem 2.18. The distance version of $F-$ index of $G_1 \oplus G_2$ is given by $2 \times DDF(G_1 \oplus G_2)$

$$
\begin{aligned}
= & \; 4\Big(n_2^2m_2 + 2F(G_2) - 2n_2M_1(G_2)\Big)\Big[(n_1-1)M_1(G_1) - F(G_1) + M_1(G_1)\Big] \\
+ & \; 4\Big(n_1^2m_1 + 2F(G_1) - 2n_1M_1(G_1)\Big)\Big[(n_2-1)M_1(G_2) - F(G_2) + M_1(G_2)\Big] \\
+ & \; 2n_1^2(2\overline{m}_1 + n_1)F(G_2) + 4n_1\Big(n_2M_1(G_2) - 2F(G_2)\Big)\Big[2\overline{m}_1(n_1-1) + 2m_1 - M_1(\overline{G}_1)\Big] \\
+ & \; 2n_2^2(2\overline{m}_2 + n_2)F(G_1) + 4n_2\Big(n_1M_1(G_1) - 2F(G_1)\Big)\Big[2\overline{m}_2(n_2-1) + 2m_2 - M_1(\overline{G}_2)\Big] \\
+ & \; 8n_2^2F(G_1)m_2 + 8n_1^2m_1F(G_2) + 16F(G_1)F(G_2) + 8n_1n_2M_1(G_1)M_1(G_2) \\
- & \; 16n_2F(G_1)M_1(G_2) + 4n_2^2\Big[(n_1-1)M_1(G_1) - F(G_1) + M_1(G_1)\Big](2\overline{m}_2 + n_2) \\
- & \; 16n_1M_1(G_1)F(G_2) + 4n_1^2\Big[(n_2-1)M_1(G_2) - F(G_2) + M_1(G_2)\Big](2\overline{m}_1 + n_1) \\
+ & \; 16\Big[(n_1-1)M_1(G_1) - F(G_1) + M_1(G_1)\Big]\Big[(n_2-1)M_1(G_2) - F(G_2) + M_1(G_2)\Big]
\end{aligned}
$$

$$
\begin{aligned}
+ & \; 8n_1n_2\Big[2\overline{m}_1(n_1-1) + 2m_1 - M_1(\overline{G}_1))(2\overline{m}_2(n_2-1) + 2m_2 - M_1(\overline{G}_2)\Big] \\
- & \; 16n_1\Big(2\overline{m}_1(n_1-1) + 2m_1 - M_1(\overline{G}_1)\Big)\Big((n_2-1)M_1(G_2) - F(G_2) + M_1(G_2)\Big)
\end{aligned}
$$

$$\begin{aligned}
&- \quad 16n_2\Big[2(2\overline{m}_2(n_2-1)+2m_2-M_1(\overline{G}_2))((n_1-1)M_1(G_1)-F(G_1)+M_1(G_1))\Big]\\
&- \quad 4n_2^3M_1(G_1)-4n_1^3M_1(G_2)-16M_1(G_1)M_1(G_2)-32n_1n_2m_1m_2\\
&+ \quad 32n_1m_1M_1(G_2)+32n_2M_1(G_1)m_2
\end{aligned}$$

Proof: Let $G = G_1 \oplus G_2$

$$\begin{aligned}
2\times DF(G_1\oplus G_2) &= \sum_{x,y\in V(G_1)}\sum_{u,v\in V(G_2)} d_G((x,u)(y,v))[d_G(x,u)^2+d_G(y,v)^2]\\
&= \sum_{x,y\in V(G_1)}\Big\{\sum_{uv\in G_2} d_G((x,u)(y,v))[d_G(x,u)^2+d_G(y,v)^2]\\
&+ \sum_{uv\notin G_2} d_G((x,u)(y,v))[d_G(x,u)^2+d_G(y,v)^2]\Big\}\\
&= \sum_{x,y\in V(G_1)}\sum_{uv\in G_2} d_G((x,u)(y,v))[d_G(x,u)^2+d_G(y,v)^2]\\
&+ \sum_{x,y\in V(G_1)}\sum_{uv\notin G_2} d_G((x,u)(y,v))[d_G(x,u)^2+d_G(y,v)^2]\\
&= \sum_{xy\in G_1}\sum_{uv\in G_2} d_G((x,u)(y,v))[d_G(x,u)^2+d_G(y,v)^2]\\
&+ \sum_{xy\notin G_1}\sum_{uv\in G_2} d_G((x,u)(y,v))[d_G(x,u)^2+d_G(y,v)^2]\\
&+ \sum_{xy\in G_1}\sum_{uv\notin G_2} d_G((x,u)(y,v))[d_G(x,u)^2+d_G(y,v)^2]\\
&+ \sum_{xy\notin G_1}\sum_{uv\notin G_2} d_G((x,u)(y,v))[d_G(x,u)^2+d_G(y,v)^2]\\
&= B_3+B_1+B_2+B_4,
\end{aligned}$$

where B_3, B_1, B_2 and B_4 are terms of the above sums taken in order. We compute B_1, B_2, B_3 and B_4 separately.

$$\begin{aligned}
B_1 &= \sum_{xy\notin G_1}\sum_{uv\in G_2} d_G((x,u)(y,v))[d_G(x,u)^2+d_G(y,v)^2]\\
&= \sum_{xy\notin G_1}\sum_{uv\in G_2} [d_G(x,u)^2+d_G(y,v)^2]\\
&= \sum_{xy\notin G_1}\sum_{uv\in G_2}\Big[\Big(n_2d_{G_1}(x)+n_1d_{G_2}(u)-2d_{G_1}(x)d_{G_2}(u)\Big)^2+\Big(n_2d_{G_1}(y)+n_1d_{G_2}(v)\\
&- \quad 2d_{G_1}(y)d_{G_2}(v)\Big)^2\Big]
\end{aligned}$$

$$= \sum_{xy\in G_1}\sum_{uv\notin G_2}\Big[n_2^2d_{G_1}^2(x)+n_1^2d_{G_2}^2(u)+4d_{G_1}^2(x)d_{G_2}^2(u)+2n_1n_2d_{G_1}(x)d_{G_2}(u)$$

$$
\begin{aligned}
&- \ 4n_1d_{G_1}(x)d^2_{G_2}(u) - 4n_2d^2_{G_1}(x)d_{G_2}(u) + n_2^2d^2_{G_1}(y) + n_1^2d^2_{G_2}(v) + 4d^2_{G_1}(y)d^2_{G_2}(v)\\
&+ \ 2n_1n_2d_{G_1}(y)d_{G_2}(v) - 4n_1d_{G_1}(y)d^2_{G_2}(v) - 4n_2d^2_{G_1}(y)d_{G_2}(v)\Big]\\
&= \ n_2^2\sum_{xy\notin G_1}\sum_{uv\in G_2}\Big(d^2_{G_1}(x) + d^2_{G_1}(y)\Big) + n_1^2\sum_{xy\notin G_1}\sum_{uv\in G_2}[d^2_{G_2}(u) + d^2_{G_2}(v)]\\
&+ \ 4\sum_{xy\notin G_1}\sum_{uv\in G_2}[d^2_{G_1}(x)d^2_{G_2}(u) + d^2_{G_1}(y)d^2_{G_2}(v)]\\
&+ \ 2n_1n_2\sum_{xy\notin G_1}\sum_{uv\in G_2}[d_{G_1}(x)d_{G_2}(u) + d_{G_1}(y)d_{G_2}(v)]\\
&- \ 4n_1\sum_{xy\notin G_1}\sum_{uv\in G_2}[d_{G_1}(x)d^2_{G_2}(u) + d^2_{G_2}(v)d_{G_1}(y)]\\
&- \ 4n_2\sum_{xy\notin G_1}\sum_{uv\in G_2}[d^2_{G_1}(x)d_{G_2}(u) + d^2_{G_1}(y)d_{G_2}(v)]\\
&= \ n_2^2\sum_{xy\notin G_1}\Big(d^2_{G_1}(x) + d^2_{G_1}(y)\Big)\sum_{uv\in G_2}1 + n_1^2\sum_{xy\notin G_1}1\sum_{uv\in G_2}[d^2_{G_2}(u) + d^2_{G_2}(v)]\\
&+ \ 4\sum_{xy\notin G_1}[d^2_{G_1}(x) + d^2_{G_1}(y)]\sum_{u\in G_2}d^3_{G_2}(u) + 2n_1n_2\sum_{xy\notin G_1}[d_{G_1}(x) + d_{G_1}(y)]\sum_{u\in V(G_1)}d^2_{G_2}(u)\\
&- \ 4n_1\sum_{xy\notin G_1}[d_{G_1}(x) + d_{G_1}(y)]\sum_{u\in V(G_1)}d^3_{G_2}(u) - 4n_2\sum_{xy\notin G_1}[d^2_{G_1}(x) + d^2_{G_1}(y)]\sum_{u\in V(G_1)}d^2_{G_2}(u)\\
&= \ n_2^2(2[(n_1-1)M_1(G_1) - F(G_1) + M_1(G_1)])(2m_2) + n_1^2(2\overline{m}_1 + n_1)2F(G_2)\\
&+ \ 8[(n_1-1)M_1(G_1) - F(G_1) + M_1(G_1)]F(G_2) + 2n_1n_2\Big(2[2\overline{m}_1(n_1-1)\\
&+ \ 2m_1 - M_1(\overline{G}_1)]\Big)M_1(G_2) - 4n_1(2[2\overline{m}_1(n_1-1) + 2m_1 - M_1(\overline{G}_1)])F(G_2)\\
&- \ 4n_2(2[(n_1-1)M_1(G_1) - F(G_1) + M_1(G_1)])(M_1(G_2))
\end{aligned}
$$

$$
\begin{aligned}
B_2 &= \sum_{xy\in G_1}\sum_{uv\notin G_2}d_G((x,u)(y,v))[d_G(x,u)^2 + d_G(y,v)^2]\\
&= \sum_{xy\in G_1}\sum_{uv\notin G_2}[d_G(x,u)^2 + d_G(y,v)^2]\\
&= \sum_{xy\in G_1}\sum_{uv\notin G_2}\Big[\Big(n_2d_{G_1}(x) + n_1d_{G_2}(u) - 2d_{G_1}(x)d_{G_2}(u)\Big)^2 + \Big(n_2d_{G_1}(y) + n_1d_{G_2}(v)\\
&- \ 2d_{G_1}(y)d_{G_2}(v)\Big)^2\Big]\\
&= \sum_{xy\in G_1}\sum_{uv\notin G_2}\Big[n_2^2d^2_{G_1}(x) + n_1^2d^2_{G_2}(u) + 4d^2_{G_1}(x)d^2_{G_2}(u) + 2n_1n_2d_{G_1}(x)d_{G_2}(u)\\
&- \ 4n_1d_{G_1}(x)d^2_{G_2}(u) - 4n_2d^2_{G_1}(x)d_{G_2}(u) + n_2^2d^2_{G_1}(y) + n_1^2d^2_{G_2}(v) + 4d^2_{G_1}(y)d^2_{G_2}(v)\\
&+ \ 2n_1n_2d_{G_1}(y)d_{G_2}(v) - 4n_1d_{G_1}(y)d^2_{G_2}(v) - 4n_2d^2_{G_1}(y)d_{G_2}(v)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= n_2^2 \sum_{xy\in G_1} \sum_{uv\notin G_2} \Big(d^2_{G_1}(x) + d^2_{G_1}(y)\Big) + n_1^2 \sum_{xy\in G_1} \sum_{uv\notin G_2} [d^2_{G_2}(u) + d^2_{G_2}(v)] \\
&+ 4 \sum_{xy\in G_1} \sum_{uv\notin G_2} [d^2_{G_1}(x)d^2_{G_2}(u) + d^2_{G_1}(y)d^2_{G_2}(v)] \\
&+ 2n_1n_2 \sum_{xy\in G_1} \sum_{uv\notin G_2} [d_{G_1}(x)d_{G_2}(u) + d_{G_1}(y)d_{G_2}(v)] \\
&- 4n_1 \sum_{xy\in G_1} \sum_{uv\notin G_2} [d_{G_1}(x)d^2_{G_2}(u) + d^2_{G_2}(v)d_{G_1}(y)] \\
&- 4n_2 \sum_{xy\in G_1)} \sum_{uv\notin G_2} [d^2_{G_1}(x)d_{G_2}(u) + d^2_{G_1}(y)d_{G_2}(v)] \\
&= n_2^2 \sum_{xy\in G_1} \Big(d^2_{G_1}(x) + d^2_{G_1}(y)\Big) \sum_{uv\notin G_2} 1 + n_1^2 \sum_{xy\in G_1} 1 \sum_{uv\notin G_2} [d^2_{G_2}(u) + d^2_{G_2}(v)] \\
&+ 4 \sum_{xy\in G_1} d^2_{G_1}(x) \sum_{uv\notin G_2} [d^2_{G_2}(u) + d^2_{G_2}(v)] + 2n_1n_2 \sum_{xy\in G_1} d_{G_1}(x) \sum_{uv\notin G_2} [d_{G_2}(u) + d_{G_2}(v)] \\
&- 4n_1 \sum_{xy\in G_1} d_{G_1}(x) \sum_{uv\notin G_2} [d^2_{G_2}(u) + d^2_{G_2}(v)] - 4n_2 \sum_{xy\in G_1} d^2_{G_1}(x) \sum_{uv\notin G_2} [d_{G_2}(u) + d_{G_2}(v)] \\
&= n_2^2 \sum_{xy\in G_1} \Big(d^2_{G_1}(x) + d^2_{G_1}(y)\Big)\Big(\sum_{uv\notin G_2} 1\Big) \\
&+ n_1^2 \sum_{xy\in G_1} 1 \sum_{uv\notin G_2} [d^2_{G_2}(u) + d^2_{G_2}(v)] + 4 \sum_{x\in V(G_1)} d^3_{G_1}(x) \sum_{uv\notin G_2} [d^2_{G_2}(u) + d^2_{G_2}(v)] \\
&+ 2n_1n_2 \sum_{x\in V(G_1)} d^2_{G_1}(x) \sum_{uv\notin G_2} [d_{G_2}(u) + d_{G_2}(v)] - 4n_1 \sum_{x\in V(G_1)} d^2_{G_1}(x) \sum_{uv\notin G_2} [d^2_{G_2}(u) + d^2_{G_2}(v)] \\
&- 4n_2 \sum_{x\in V(G_1)} d^3_{G_1}(x) \sum_{uv\notin G_2} [d_{G_2}(u) + d_{G_2}(v)] \\
&= n_2^2(2F(G_1))(2\overline{m}_2 + n_2) + n_1^2(2m_1)\Big(2[(n_2-1)M_1(G_2) - F(G_2) + M_1(G_2)]\Big) \\
&+ 2\times 4[(n_2-1)M_1(G_2) - F(G_2) + M_1(G_2)]F(G_1) \\
&+ 2n_1n_2M_1(G_1)2\Big[2\overline{m}_2(n_2-1) + 2m_2 - M_1(\overline{G_2})\Big] \\
&- 4n_1M_1(G_1)\Big(2[(n_2-1)M_1(G_2) - F(G_2) + M_1(G_2)]\Big) \\
&- 4n_2F(G_1)\Big(2[2\overline{m}_2(n_2-1) + 2m_2 - M_1(\overline{G}_2)]\Big)
\end{aligned}
$$

$$
\begin{aligned}
B_3 &= \sum_{xy\in G_1)} \sum_{uv\in G_2)} d_G((x,u)(y,v))[d_G(x,u)^2 + d_G(y,v)^2] \\
&= 2 \sum_{xy\in G_1} \sum_{uv\in G_2} [d_G(x,u)^2 + d_G(y,v)^2] \\
&= 2 \sum_{xy\in G_1} \sum_{uv\in G_2} \Big[\Big(n_2d_{G_1}(x) + n_1d_{G_2}(u) - 2d_{G_1}(x)d_{G_2}(u)\Big)^2 + \Big(n_2d_{G_1}(y) + n_1d_{G_2}(v)
\end{aligned}
$$

$$
\begin{aligned}
&- \quad 2d_{G_1}(y)d_{G_2}(v)\Big)^2\Big] \\
&= \quad 2\sum_{xy\in G_1}\sum_{uv\in G_2}\Big[n_2^2 d_{G_1}^2(x) + n_1^2 d_{G_2}^2(u) + 4d_{G_1}^2(x)d_{G_2}^2(u) + 2n_1n_2 d_{G_1}(x)d_{G_2}(u) \\
&- \quad 4n_1 d_{G_1}(x)d_{G_2}^2(u) - 4n_2 d_{G_1}^2(x)d_{G_2}(u) + n_2^2 d_{G_1}^2(y) + n_1^2 d_{G_2}^2(v) + 4d_{G_1}^2(y)d_{G_2}^2(v) \\
&+ \quad 2n_1n_2 d_{G_1}(y)d_{G_2}(v) - 4n_1 d_{G_1}(y)d_{G_2}^2(v) - 4n_2 d_{G_1}^2(y)d_{G_2}(v)\Big] \\
&= \quad 2\Big[n_2^2\sum_{xy\in E(G_1)}\sum_{uv\in E(G_2)}\Big(d_{G_1}^2(x) + d_{G_1}^2(y)\Big) + n_1^2\sum_{xy\in E(G_1)}\sum_{uv\in E(G_2)}[d_{G_2}^2(u) + d_{G_2}^2(v)] \\
&+ \quad 4\sum_{xy\in E(G_1)}\sum_{uv\in E(G_2)}[d_{G_1}^2(x)d_{G_2}^2(u) + d_{G_1}^2(y)d_{G_2}^2(v)] \\
&+ \quad 2n_1n_2\sum_{xy\in E(G_1)}\sum_{uv\in E(G_2)}[d_{G_1}(x)d_{G_2}(u) + d_{G_1}(y)d_{G_2}(v)] \\
&- \quad 4n_1\sum_{xy\in E(G_1)}\sum_{uv\in E(G_2)}[d_{G_1}(x)d_{G_2}^2(u) + d_{G_2}^2(v)d_{G_1}(y)] \\
&- \quad 4n_2\sum_{xy\in E(G_1)}\sum_{uv\in E(G_2)}[d_{G_1}^2(x)d_{G_2}(u) + d_{G_1}^2(y)d_{G_2}(v)]\Big] \\
&= \quad 2\Big[n_2^2\sum_{xy\in E(G_1)}\Big(d_{G_1}^2(x) + d_{G_1}^2(y)\Big)\sum_{uv\in E(G_2)}1 + n_1^2\sum_{xy\in G_1}1\sum_{uv\in G_2}[d_{G_2}^2(u) + d_{G_2}^2(v)] \\
&+ \quad 4\sum_{xy\in G_1}d_{G_1}^2(x)\sum_{uv\in G_2}[d_{G_2}^2(u) + d_{G_2}^2(v)] + 2n_1n_2\sum_{xy\in G_1}[d_{G_1}(x) + d_{G_1}(y)]\sum_{u\in G_2}d_{G_2}^2(u) \\
&- \quad 4n_1\sum_{xy\in G_1}[d_{G_1}(x) + d_{G_1}(y)]\sum_{u\in G_2}d_{G_2}^3(u) - 4n_2\sum_{xy\in G_1}[d_{G_1}^2(x) + d_{G_1}^2(y)]\sum_{u\in G_2}d_{G_2}^2(u)\Big] \\
&= \quad 2\Big[n_2^2(2F(G_1))2m_2 + n_1^2(2m_1)(2F(G_2)) + 2\times 4F(G_1)F(G_2) + 2n_1n_2(2M_1(G_1))M_1(G_2) \\
&- \quad 4n_1(2M_1(G_1))(F(G_2)) - 4n_2(2F(G_1))(M_1(G_2))\Big] \\
B_4 &= \quad \sum_{xy\notin G_1}\sum_{uv\notin G_2}d_G((x,u)(y,v))[d_G(x,u)^2 + d_G(y,v)^2] \\
&= \quad \sum_{xy\notin G_1}\Big\{\sum_{uv\notin G_2,u\neq v}d_G((x,u)(y,v))[d_G(x,u)^2 + d_G(y,v)^2] \\
&+ \quad \sum_{uv\notin G_2,u=v}d_G((x,u)(y,v))d_G((x,u)(y,v))[d_G(x,u)^2 + d_G(y,v)^2]\Big\} \\
&= \quad \sum_{xy\notin G_1}\sum_{uv\notin G_2,u\neq v}d_G((x,u)(y,v))[d_G(x,u)^2 + d_G(y,v)^2] \\
&+ \quad \sum_{xy\notin G_1}\sum_{uv\notin G_2,u=v}d_G((x,u)(y,v))[d_G(x,u)^2 + d_G(y,v)^2]
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{xy\notin G_1, x\neq y} \sum_{uv\notin G_2, u\neq v} d_G((x,u)(y,v))[d_G(x,u)^2+d_G(y,v)^2] \\
&+ \sum_{xy\notin G_1, x=y} \sum_{uv\notin G_2, u\neq v} d_G((x,u)(y,v))[d_G(x,u)^2+d_G(y,v)^2] \\
&+ \sum_{xy\notin G_1, x\neq y} \sum_{uv\notin G_2, u=v} d_G((x,u)(y,v))[d_G(x,u)^2+d_G(y,v)^2] \\
&+ \sum_{xy\notin G_1, x=y} \sum_{uv\notin G_2, u=v} d_G((x,u)(y,v))[d_G(x,u)^2+d_G(y,v)^2] \\
&= 2\sum_{xy\notin G_1, x\neq y} \sum_{uv\notin G_2, u\neq v} [d_G(x,u)^2+d_G(y,v)^2] \\
&+ 2\sum_{xy\notin G_1, x=y} \sum_{uv\notin G_2, u\neq v} [d_G(x,u)^2+d_G(y,v)^2] \\
&+ 2\sum_{xy\notin G_1, x\neq y} \sum_{uv\notin G_2, u=v} [d_G(x,u)^2+d_G(y,v)^2] + 0 \\
&= 2\sum_{xy\notin G_1, x\neq y} \sum_{uv\notin G_2, u\neq v} [d_G(x,u)^2+d_G(y,v)^2] \\
&+ 2\sum_{xy\notin G_1, x=y} \sum_{uv\notin G_2, u\neq v} [d_G(x,u)^2+d_G(y,v)^2] \\
&+ 2\sum_{xy\notin G_1, x\neq y} \sum_{uv\notin G_2, u=v} [d_G(x,u)^2+d_G(y,v)^2] \\
&+ 2\sum_{xy\notin G_1, x=y} \sum_{uv\notin G_2, u=v} [d_G(x,u)^2+d_G(y,v)^2] \\
&- 2\sum_{xy\notin G_1, x=y} \sum_{uv\notin G_2, u=v} [d_G(x,u)^2+d_G(y,v)^2] \\
&= 2\sum_{xy\notin G_1} \sum_{uv\notin G_2} [d_G(x,u)^2+d_G(y,v)^2] \\
&- 2\sum_{xy\notin G_1, x=y} \sum_{uv\notin G_2, u=v} [d_G(x,u)^2+d_G(y,v)^2]
\end{aligned}
$$

$B_4 = 2B_5 - 2B_6$, where B_5 and B_6 are terms of the above sums taken in order.

$$
\begin{aligned}
B_5 &= \sum_{xy\notin G_1} \sum_{uv\notin G_2} [d_G(x,u)^2+d_G(y,v)^2] \\
&= \sum_{xy\notin G_1} \sum_{uv\notin (G_2} \Big[\Big(n_2 d_{G_1}(x)+n_1 d_{G_2}(u)-2d_{G_1}(x)d_{G_2}(u)\Big)^2+\Big(n_2 d_{G_1}(y)+n_1 d_{G_2}(v) \\
&- 2d_{G_1}(y)d_{G_2}(v)\Big)^2\Big] \\
&= \sum_{xy\notin G_1} \sum_{uv\notin G_2} \Big[n_2^2 d_{G_1}^2(x)+n_1^2 d_{G_2}^2(u)+4d_{G_1}^2(x)d_{G_2}^2(u)+2n_1n_2 d_{G_1}(x)d_{G_2}(u) \\
&- 4n_1 d_{G_1}(x)d_{G_2}^2(u)-4n_2 d_{G_1}^2(x)d_{G_2}(u)+n_2^2 d_{G_1}^2(y)+n_1^2 d_{G_2}^2(v)+4d_{G_1}^2(y)d_{G_2}^2(v)
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad 2n_1n_2d_{G_1}(y)d_{G_2}(v) - 4n_1d_{G_1}(y)d^2_{G_2}(v) - 4n_2d^2_{G_1}(y)d_{G_2}(v)\Big] \\
&= \quad n_2^2 \sum_{xy\notin G_1} \sum_{uv\notin G_2} \Big(d^2_{G_1}(x) + d^2_{G_1}(y)\Big) + n_1^2 \sum_{xy\notin G_1} \sum_{uv\notin G_2} [d^2_{G_2}(u) + d^2_{G_2}(v)] \\
&+ \quad 4 \sum_{xy\notin G_1} \sum_{uv\notin G_2)} [d^2_{G_1}(x)d^2_{G_2}(u) + d^2_{G_1}(y)d^2_{G_2}(v)] \\
&+ \quad 2n_1n_2 \sum_{xy\notin G_1} \sum_{uv\notin G_2} [d_{G_1}(x)d_{G_2}(u) + d_{G_1}(y)d_{G_2}(v)] \\
&- \quad 4n_1 \sum_{xy\notin G_1} \sum_{uv\notin G_2} [d_{G_1}(x)d^2_{G_2}(u) + d^2_{G_2}(v)d_{G_1}(y)] \\
&- \quad 4n_2 \sum_{xy\notin G_1} \sum_{uv\notin G_2} [d^2_{G_1}(x)d_{G_2}(u) + d^2_{G_1}(y)d_{G_2}(v)] \\
&= \quad n_2^2 \sum_{xy\notin G_1} \Big(d^2_{G_1}(x) + d^2_{G_1}(y)\Big) \sum_{uv\notin G_2} 1 + n_1^2 \sum_{xy\notin G_1} 1 \sum_{uv\notin G_2} [d^2_{G_2}(u) + d^2_{G_2}(v)] \\
&+ \quad 4\times 4 \sum_{xy\notin G_1} d^2_{G_1}(x) \sum_{uv\notin G_2)} d^2_{G_2}(u) + \sum_{xy\notin G_1} d^2_{G_1}(y) \sum_{uv\notin G_2)} d^2_{G_2}(v) \\
&+ \quad 2n_1n_2\Big[\sum_{xy\notin G_1} d_{G_1}(x) \sum_{uv\notin G_2} d_{G_2}(u) + \sum_{xy\notin G_1} d_{G_1}(y) \sum_{uv\notin G_2} d_{G_2}(v)\Big] \\
&- \quad 2\times 4n_1\Big[\sum_{xy\notin G_1} d_{G_1}(x) \sum_{uv\notin G_2} d^2_{G_2}(u) + \sum_{uv\notin G_2} d^2_{G_2}(v) \sum_{xy\notin G_1} d_{G_1}(y)\Big] \\
&- \quad 2\times 4n_2\Big[\sum_{xy\notin G_1} d^2_{G_1}(x) \sum_{uv\notin G_2} d_{G_2}(u) + \sum_{xy\notin G_1} d^2_{G_1}(y) \sum_{uv\notin G_2} d_{G_2}(v)\Big] \\
&= \quad n_2^2\, 2\Big[(n_1-1)M_1(G_1) - F(G_1) + M_1(G_1)\Big](2\overline{m}_2 + n_2) \\
&+ \quad n_1^2\, 2\Big[(n_2-1)M_1(G_2) - F(G_2) + M_1(G_2)\Big](2\overline{m}_1 + n_1) \\
&+ \quad 2\times 4\Big[(n_1-1)M_1(G_1) - F(G_1) + M_1(G_1)\Big]\Big[(n_2-1)M_1(G_2) - F(G_2) + M_1(G_2)\Big] \\
&+ \quad 2n_1n_2\Big[2(2\overline{m}_1(n_1-1) + 2m_1 - M_1(\overline{G}_1))(2\overline{m}_2(n_2-1) + 2m_2 - M_1(\overline{G}_2))\Big] \\
&- \quad 4n_1\Big[2(2\overline{m}_1(n_1-1) + 2m_1 - M_1(\overline{G}_1))((n_2-1)M_1(G_2) - F(G_2) + M_1(G_2))\Big] \\
&- \quad 4n_2\Big[2(2\overline{m}_2(n_2-1) + 2m_2 - M_1(\overline{G}_2))((n_1-1)M_1(G_1) - F(G_1) + M_1(G_1))\Big] \\
B_6 &= \quad \sum_{xy\notin G_1,\, x=y} \sum_{uv\notin G_2,\, u=v} [d_G(x,u)^2 + d_G(y,v)^2] \\
&= \quad \sum_{xy\notin G_1,\, x=y} \sum_{uv\notin (G_2,\, u=v} \Big[\Big(n_2d_{G_1}(x) + n_1d_{G_2}(u) - 2d_{G_1}(x)d_{G_2}(u)\Big)^2
\end{aligned}
$$

$$
\begin{aligned}
&+ \Big(n_2 d_{G_1}(y) + n_1 d_{G_2}(v) - 2d_{G_1}(y) d_{G_2}(v)\Big)^2\Big] \\
&= \sum_{xy\notin G_1,\, x=y} \sum_{uv\notin G_2,\, u=v} \Big[n_2^2 d_{G_1}^2(x) + n_1^2 d_{G_2}^2(u) + 4d_{G_1}^2(x) d_{G_2}^2(u) + 2n_1 n_2 d_{G_1}(x) d_{G_2}(u) \\
&- 4n_1 d_{G_1}(x) d_{G_2}^2(u) - 4n_2 d_{G_1}^2(x) d_{G_2}(u) + n_2^2 d_{G_1}^2(y) + n_1^2 d_{G_2}^2(v) + 4d_{G_1}^2(y) d_{G_2}^2(v) \\
&+ 2n_1 n_2 d_{G_1}(y) d_{G_2}(v) - 4n_1 d_{G_1}(y) d_{G_2}^2(v) - 4n_2 d_{G_1}^2(y) d_{G_2}(v)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= n_2^2 \sum_{xy\notin G_1,\, x=y} \sum_{uv\notin G_2,\, u=v} \Big(d_{G_1}^2(x) + d_{G_1}^2(y)\Big) + n_1^2 \sum_{xy\notin G_1,\, x=y} \sum_{uv\notin G_2,\, u=v} [d_{G_2}^2(u) + d_{G_2}^2(v)] \\
&+ 4 \sum_{xy\notin G_1,\, x=y} \sum_{uv\notin G_2,\, u=v} [d_{G_1}^2(x) d_{G_2}^2(u) + d_{G_1}^2(y) d_{G_2}^2(v)] \\
&+ 2n_1 n_2 \sum_{xy\notin G_1,\, x=y} \sum_{uv\notin G_2,\, u=v} [d_{G_1}(x) d_{G_2}(u) + d_{G_1}(y) d_{G_2}(v)] \\
&- 4n_1 \sum_{xy\notin G_1,\, x=y} \sum_{uv\notin G_2,\, u=v} [d_{G_1}(x) d_{G_2}^2(u) + d_{G_2}^2(v) d_{G_1}(y)] \\
&- 4n_2 \sum_{xy\notin G_1,\, x=y} \sum_{uv\notin G_2,\, u=v} [d_{G_1}^2(x) d_{G_2}(u) + d_{G_1}^2(y) d_{G_2}(v)] \\
&= n_2^2 \sum_{xy\notin G_1,\, x=y} \Big(d_{G_1}^2(x) + d_{G_1}^2(y)\Big) \sum_{uv\notin G_2,\, u=v} 1 + n_1^2 \sum_{xy\notin G_1,\, x=y} 1 \sum_{uv\notin G_2,\, u=v} [d_{G_2}^2(u) + d_{G_2}^2(v)] \\
&+ 4 \sum_{xy\notin G_1,\, x=y} d_{G_1}^2(x) \sum_{uv\notin G_2,\, u=v} d_{G_2}^2(u) + \sum_{xy\notin G_1,\, x=y} d_{G_1}^2(y) \sum_{uv\notin G_2,,\, u=v} d_{G_2}^2(v) \\
&+ 2n_1 n_2 \Big[\sum_{xy\notin G_1,\, x=y} d_{G_1}(x) \sum_{uv\notin G_2,\, u=v} d_{G_2}(u) + \sum_{xy\notin G_1,\, x=y} d_{G_1}(y) \sum_{uv\notin G_2,\, u=v} d_{G_2}(v)\Big] \\
&- 4n_1 \Big[\sum_{xy\notin G_1,\, x=y} d_{G_1}(x) \sum_{uv\notin G_2,\, u=v} d_{G_2}^2(u) + \sum_{uv\notin G_2,\, u=v} d_{G_2}^2(v) \sum_{xy\notin G_1,\, x=y} d_{G_1}(y)\Big] \\
&- 4n_2 \Big[\sum_{xy\notin G_1,\, x=y} d_{G_1}^2(x) \sum_{uv\notin G_2,\, u=v} d_{G_2}(u) + \sum_{xy\notin G_1,\, x=y} d_{G_1}^2(y) \sum_{uv\notin G_2,\, u=v} d_{G_2}(v)\Big] \\
&= n_2^2 (2M_1(G_1)) n_2 + n_1^2 (2M_1(G_2)) n_1 + 2 \times 4M_1(G_1) M_1(G_2) + 2n_1 n_2\, 2(2m_1)(2m_2) \\
&- 4n_1\, 2(2m_1) M_1(G_2) - 4n_2\, 2M_1(G_1)(2m_2)
\end{aligned}
$$

$$\therefore 2 \times DF(G_1 \oplus G_2) = B_3 + B_1 + B_2 + B_4 = B_3 + B_1 + B_2 + 2B_5 - 2B_6 \tag{3}$$

Substituting B_3, B_1, B_2, B_5 and B_6 in (3), the desired result follows after simple calculation. ∎

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